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Upper Semicontinuity of the Attractor with Respect to Parameter Dependent Delays

GWENDOLEN HINES*

Department of Mathematics and Statistics, University of Nebraska, Lincoln, Nebraska 68588-0323

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1. INTRODUCTION

In this paper, we investigate the dependence of the asymptotic dynamics on the delay for a certain class of delay-differential equations. This class is given by equations of the form

$$\dot{x}(t) = h\left(\int_{-\infty}^{0} p_1(x(t+\theta))\, d\eta_{1\lambda}(\theta), ..., \int_{-\infty}^{0} p_m(x(t+\theta))\, d\eta_{m\lambda}(\theta)\right) \quad (1.0)$$

where $x(t) \in \mathbb{R}^n$ and the functions $\eta_{j\lambda}: (-\infty, 0] \to \mathbb{R}^n$ have bounded variation and depend on the parameter λ. We also assume that h and $p_1, ..., p_m$ are locally Lipschitz continuous. If the support of each $d\eta_{j\lambda}$ is compact, we say that (1.0) is a *finite delay problem*. Otherwise, it is an *infinite delay problem*. For each problem, we define $r(\lambda)$ so that the interval $[-r(\lambda), 0]$ is the smallest interval which contains the support of each of the measures $d\eta_{j\lambda}$; of course, $r(\lambda)$ may be infinite. Now the problem (1.0) can be written as a retarded functional differential equation if we define the function $x_t: [-r(\lambda), 0] \to \mathbb{R}^n$ by $x_t(\theta) = x(t+\theta)$. Then the right hand side can be written as a functional on a suitable Banach space B, which we refer to as the *phase space*.

$$\dot{x}(t) = h\left(\int_{-\infty}^{0} p_1(x_t(\theta))\, d\eta_{1\lambda}(\theta), ..., \int_{-\infty}^{0} p_m(x_t(\theta))\, d\eta_{m\lambda}(\theta)\right) := f_\lambda(x_t) \quad (1.1)_\lambda$$

Problems of this form arise in many applications. The equation

$$\dot{x}(t) = -\int_{-\infty}^{0} g(\theta)\, p(x(t+\theta))\, d\theta, \quad (1.2)$$

where g is continuous and has compact support in an interval $[-r, 0]$, has been encountered by Ergen [1954] in the study of circulating fuel reactors.

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Here r is the time it takes for a particle to go through the reactor. Levin and Nohel [1964] have studied the behaviour of solutions of (1.2) as $t \to \infty$.

Equations of the form

$$\dot{x}(t) = x(t) h\left(\int_{-\infty}^{0} p_1(x(t+\theta))\, g_1(\theta)\, d\theta, \int_{-\infty}^{0} p_2(x(t+\theta))\, g_2(\theta)\, d\theta\right) \tag{1.3}$$

are common in economic and biological models. For example, Bélair and Mackey [1989] use the equation

$$\dot{P}(t) = P(t) f\left(D\left(\int_{-\infty}^{0} P(t+\theta)\, g_1(\theta)\, d\theta\right), S\left(\int_{-\infty}^{0} P(t+\theta)\, g_2(\theta)\, d\theta\right)\right) \tag{1.4}$$

to model relative variations in market price $P(t)$. Here D and S are demand and supply functions. They study the stability of the equilibrium as parameters are varied. The equation

$$\dot{N}(t) = bN(t) - aN^2(t) - d\left(\int_{-\infty}^{0} N(t+\theta)\, g(\theta)\, d\theta\right)^2 N(t) \tag{1.5}$$

has been suggested to study the growth of a single species population in which the growth rate responses are related to accumulating environmental intoxicants which in turn are related to the past population (see Borsellino and Torre [1974] and Cushing [1977]). Cohen, Coutsias, and Neu [1979] have discussed the asymptotic behaviour of solutions of (1.5) for a particular choice of the "memory function" g. Cushing [1977] has discussed stability of equilibria and existence of periodic solutions for equations of the form (1.3).

In the above examples, one must usually select a particular memory function or "kernel", g, in order to discuss the asymptotic behaviour of solutions. Two factors are important in the selection, accuracy of the model and relative ease of the mathematical analysis. It is common to choose a memory function of the form

$$g^N(\theta) = \sum_{n=0}^{N} a_n\, |\theta|^n\, e^{\theta}, \tag{1.6}$$

for example, Cohen, Coutsias and Neu choose the memory function $g(\theta) = -\theta e^{\theta}$ and Belair and Mackey consider the memory function $g(\theta) = e^{\theta}$. Since the functions $\{e^{\theta}, \theta e^{\theta}, \theta^2 e^{\theta}, \ldots\}$ are dense in $L^1((-\infty, 0])$, there is some hope that such a model will provide a good approximation to the true model. The advantage in using this approximation is that, with the memory function (1.6), the functional differential equation reduces to a system of ordinary differential equations (see Busenberg and Travis [1982a] ,[1982b] and Cohen, Coutsias and Neu [1979]). For this reason, memory functions of this form are called *reducible*. The analysis of the new ODE is often much simpler than the analysis of the original FDE.

This leads us to the following question. How do the asymptotic dynamics dependon the memory function? In this paper, we will address this question. In particular, we consider the situation where it is known that the problem $(1.1)_0$ admits a global attractor, $\mathscr{A}_0$. We will give conditions on the right-hand side under which, for some $\bar{\lambda}$, the problem $(1.1)_\lambda$ also admits an attractor, $\mathscr{A}_\lambda$, for all $\lambda \leqslant \bar{\lambda}$ and the family of attractors is upper semicontinuous at $\lambda = 0$. As might be expected, if the nonlinearities, f_λ, satisfy $f_\lambda \to f_0$ uniformly in bounded sets of B, then we will have existence and upper semicontinuity of the attractors $\{\mathscr{A}_\lambda\}$. In many examples where the measures $d\eta_{j\lambda}$ satisfy $d\eta_{j\lambda}(\theta) = g_{j\lambda}(\theta)\, d\theta$ for continuous functions $g_{j\lambda}$, we will be able to verify this convergence. We would also like to consider the case, however, in which the functions $\eta_{j\lambda}$ are discontinuous; that is, when the problem $(1.1)_\lambda$ has discrete delays. For example, we would like to be able to apply our theory to the following situation in which all of the perturbed problems $(1.1)_\lambda$ have infinite delay and the limiting problem$(1.1)_0$ has a single discrete delay.

In 1948, Hutchison [1948] proposed the following model to describe the population growth of a single species with saturation effect.

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t-T)}{K}\right). \tag{1.7}$$

Here, N might be the size of a herbivore population which obeys a logistic growth law with growth rate r and maximum population sustainable by the environment K, but with a time lag T, where T would be the time it takes for the vegetation to recover from being grazed. This model has been used extensively in the applications to discuss single species populations and it has been shown (see Hale [1988]) that the problem admits a compact attractor in the subspace $C^+ := \{\phi \in C([-T, 0], \mathbb{R}) \colon \phi(\theta) > 0, \theta \in [-T, 0]\}$.

In 1973, however, May [1973] suggested that it would be more realistic to assume that the population depends not on the population at some particular instant in the past, but rather on an average population over all past time; that is, the term $(N(t-T))/K$ would be better modeled as $\int_{-\infty}^{t} N(\theta)\, Q(t-\theta)\, d\theta$ where $Q(t)$ has a "bump" at $-T$ and decays to 0 at $-\infty$. Then the delay equation becomes

$$\dot{N}(t) = rN(t)\left(1 - \int_{-\infty}^{t} N(\theta)\, Q(t-\theta)\, d\theta\right). \tag{1.8}$$

May suggested that a typical memory function $Q(t)$ would have the form $Q(t) = (1/KT)(t/T)\, e^{(-t/T)}$ and, in fact, he suggested that one could find a sequence of memory functions, $\{Q_n(t)\}$, with $Q_1(t) = Q(t)$ which in some sense "converge to the δ-function". In that case the models

$$\dot{N}(t) = rN(t)\left(1 - \int_{-\infty}^{t} N(\theta)\, Q_n(t-\theta)\, d\theta\right) \tag{1.8$_\mathrm{n}$}$$

would "converge in some sense to the model (1.7)".

It is probably not possible to construct such a sequence of memory functions for which the nonlinearities converge uniformly in bounded sets, however, we can construct a sequence for which the nonlinearities converge uniformly on sets Ω for which the restriction $\hat{\Omega} := \{\hat{\omega} \in C([-T-1, 0], \mathbb{R}^n) : \hat{\omega}(\theta) = \omega(\theta)$ for $\omega \in \Omega$ and $\theta \in [-T-1, 0]\}$ is equicontinuous in $C([-T-1, 0], \mathbb{R}^n)$ (see section 7). This will prove to be good enough, since, if $T_n(t)$ is the solution semigroup associated with the problem $(1.8)_n$, then, for every bounded set Γ, the set $T_n(-T-1)\Gamma$ has this property. In general, if the points of discontinuity of the functions η_{j0} are all contained in the interval $(-r, 0]$, then in many applications, we will have the convergence $f_\lambda \to f_0$ uniformly in sets Ω for which the restriction $\Omega^r := \{\omega^r \in C([-r, 0], \mathbb{R}^n) : \omega^r(\theta) = \omega(\theta)$ for $\theta \in [-r, 0]\}$ is equicontinuous in $C([-r, 0], \mathbb{R}^n)$. We will say that such a set is C_r-*equicontinuous*. If $T_\lambda(t)$ is the solution operator associated with $(1.1)_\lambda$, then for any bounded set Γ, the set $T_\lambda(r)\Gamma$ is C_r-equicontinuous.

Other authors have investigated the dependence of solutions on the kernels for certain subclasses of equation $(1.1)_\lambda$. Busenberg and Travis ([1982a], [1982b]), Dolgii and Sazhina [1985], and Busenberg and Hill [1988] have all studied the relationship between stability of the equilibrium of the linear system

$$\dot{x}(t) = \int_{-\infty}^{0} x(t+\theta)\, d\eta(\theta), \tag{1.9}$$

$(x \in \mathbb{R}^n)$ and stability of the equilibrium of the system when we approximate the kernel by a reducible kernel. In Busenberg and Travis, the authors showed that for any $\alpha \in \mathbb{R}$, there exists a system with a reducible kernel (i.e. a "reducible system") whose spectrum coincides exactly with the set of eigenvalues of (1.9) satisfying $Re\lambda_i > -\alpha$. The difference between the solutions of the original FDE and those of the reducible system decays in time like $e^{-\alpha t}$. They use this result to obtain sufficient conditions for asymptotic stability of the trivial solution of (1.9) and to analyze the local asymptotic stability of the equilibria for a certain subclass of models of the form (1.3). It is not generally possible, however, to construct the reducible system given by Busenberg and Travis since their reducible kernel depends on the eigenvalues of (1.9). Dolgii and Sazhina, [1985], give a constructive method for finding an approximate reducible system for equations with a single delay

$$\dot{x}(t) = Ax(t) + Bx(t-T) \tag{1.10}$$

($x \in \mathbb{R}^n$) and are able to exhibit a system of $n(m+1)$ ODE such that if the solutions of the ODE decay like $e^{-\alpha t}$, for large enough m and for $\alpha < 0$, then the solutions of (1.10) are asymptotically stable. Busenberg and Hill do this for the general linear equation (1.9). Their construction relies on a knowledge of the measure $d\eta$ itself and not on a knowledge of the eigenvalues.

The effect of perturbing the kernels for scalar FDE with kernels of the form (1.6) has been considered by Farkas and Stépán [1992] in the context of robustness of Hopf bifurcations. They show that if the unperturbed equation undergoes a Hopf bifurcation as a certain parameter is varied, then for small perturbations of a certain class, the perturbed problem also undergoes a Hopf bifurcation.

Before proceeding, we must state some results from the study of dissipative systems which we will need in the sequel. These results can be found in Hale [1988]. We begin with some definitions. Suppose B is a Banach space and $T(t)$ is a semigroup on B. An invariant set $\mathscr{A} \subset B$ is said to be a *global attractor* if $\mathscr{A}$ is a maximal compact invariant set which attracts each bounded set, Γ, of B in the sense that $\delta(T(t)\Gamma, \mathscr{A}) \to 0$ as $t \to \infty$ where, for any two sets A and B, $\delta(A, B) := \sup_{\phi \in \mathscr{A}_\lambda} \inf_{\psi \in B} \operatorname{dist}(\psi, \phi)$. A sequence of sets $\{\mathscr{A}_\lambda\}$ is said to be *upper semicontinuous at* $\lambda = 0$ if $\delta(\mathscr{A}_\lambda, \mathscr{A}_0) \to 0$ as $\lambda \to 0$. For infinite dimensional systems, it is not automatic that the orbit $T(t)\Gamma$ compactifies if it is bounded. Hence, we can't even talk about ω-limit sets for bounded orbits without further hypotheses on $T(t)$. This hypothesis will be that $T(t)$ is an α-contraction. A semigroup, $T(t)$ is said to be a *conditional* α*-contraction* if there is a continuous function $k\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $k(t) \to 0$ as $t \to \infty$, and, for each $t > 0$ and each bounded set Γ for which $\{T(s)\Gamma, 0 \leqslant s \leqslant t\}$ is bounded, we have $\alpha(T(t)\Gamma) \leqslant k(t)\,\alpha(\Gamma)$. Here α is the *Kuratowski measure of noncompactness* defined by $\alpha(\Gamma) = \inf\{d\colon \Gamma$ has a finite cover of diameter $< d\}$. $T(t)$ is an α*-contraction* if also $\{T(s)\Gamma, 0 \leqslant s \leqslant t\}$ is bounded whenever Γ is. It is then a result of Hale [1988] that if $T(t)$ is a conditional α-contraction and Γ is a bounded set for which the forward orbit $\gamma^+(\Gamma)$ is bounded, then $\omega(\Gamma)$ exists and is compact, invariant and attracts Γ. To have existence of a global attractor, then, it should be enough to have that the semigroup is a conditional α-contraction, that orbits of bounded sets are bounded and some sort of dissipation. For our purposes, we will use the following result of Hale [1988]. We say that the semigroup $T(t)$ is *bounded dissipative* if there is a bounded set $\Gamma \subset B$ that attracts each bounded set of B. If the semigroup $T(t)$ is bounded dissipative and is a conditional α-contraction then there exists a global attractor.

2. The Phase Space

The first step in discussing any delay problem is to settle on an appropriate phase space. For finite delay problems with delay r, it is natural to choose the space $C([-r, 0], \mathbb{R}^n)$. The choice of the phase space for infinite delay problems is discussed extensively in Hale and Kato [1978] and in Hino, Murakami, and Naito [1991], and in general it depends on the delay. In our problem then, the phase space will depend on the parameter and this, of course, presents special problems when trying to compare orbits for different values of the parameter. First we will discuss the choice of the phase space for infinite delay problems without parameter dependence and then we will address the issue of parameter dependence for both finite and infinite delay problems.

The above authors give a set of fundamental axioms along with certain axioms needed to develop a local theory and a global theory for infinite delay problems. Since we will be using the global theory in this paper, we present the axioms as needed for the global theory. The phase space B will be a Banach space of functions mapping $(-\infty, 0]$ into $\mathbb{R}^n$ with norm $\|\cdot\|_B$. We require that the following axioms are satisfied in B.

(A1) If x is a function mapping $(-\infty, a)$ into $\mathbb{R}^n$, $a > 0$, such that the function $x_0: (-\infty, 0] \to \mathbb{R}^n$ is in B and $x(t)$ is continuous for $t \in [0, a)$, then for every $t \in [0, a)$, the following hold

(i) $x_t \in B$

(ii) $\|x_t\|_B \leqslant K \sup_{0 \leqslant s \leqslant t} |x(s)| + M(t) \|x_0\|_B$ where $K > 0$ is a constant and $\bar{M} := \sup_t M(t) < \infty$ and for some $t_0 > 0$, $M(t_0) < 1$.

(A2) For x as in (A1), x_t is a B-valued continuous function for $t \in [0, a)$.

(A3) If ϕ is an initial condition in B, then $|\phi(0)| \leqslant H \|\phi\|_B$ for some constant $H > 0$.

(A4) All bounded continuous functions are in B.

(A5) If we define the translation operator $S(t)$ on the space $\tilde{B} := \{\phi \in B : \phi(0) = 0\}$ by

$$[S(t)\phi](\theta) = \begin{cases} 0 & -t < \theta \leqslant 0 \\ \phi(t+\theta) & \theta \leqslant -t \end{cases}$$

then $S(t)$ satisfies

(i) $S(t)$ is a bounded operator for each t.

(ii) $\|S(t)\| \to 0$ as $t \to \infty$.

These axioms are chosen so that the orbit x_t is well defined in B and so that the effect of the initial condition in the norm fades as $t \to \infty$. If $S(t)$ satisfies the axiom (A5), then it is an α-contraction. If we define the solution operator $T(t)\phi = x_t(\cdot\,;\phi)$ and the function $\tilde{\phi}(\theta) = \phi(0)$ for all $\theta \in (-\infty, 0]$, then, because of the inequality in (A1), the operator $U(t) := T(t)\phi - S(t)(\phi - \tilde{\phi})$ is conditionally compact. Therefore, $T(t)$ is a conditional α-contraction. For a more complete understanding of the choice of axioms, the reader is referred to Hale and Kato [1978] and Hino, Murakami and Naito [1991].

The phase space must be chosen appropriately for each problem. For example, a natural phase space for the problem (1.8) might be the space $B_{(8)} = \{\phi : \phi$ is measurable and $\|\phi\|_B < \infty\}$ where

$$\|\phi\|_{B_{(8)}} = |\phi(0)| + \int_{-\infty}^{0} Q(\theta)\, |\phi(\theta)|\, d\theta.$$

This space does indeed satisfy the five axioms. If we considered the other perturbed problems $(1.8)_n$ however, the space $B_{(8)}$ may not be very sensible, since an initial condition which has finite norm in $B_{(8)}$ may make the right hand side of $(1.8)_n$ infinite. It would make sense instead to consider $(1.8)_n$ in the space $B_{(8n)}$ with norm

$$\|\phi\|_{B_{(8n)}} = |\phi(0)| + \int_{-\infty}^{0} Q_n(\theta)\, |\phi(\theta)|\, d\theta.$$

It may be that $B_{(8)} \subset B_{(8n)}$ and $\|\cdot\|_{B_{(8n)}} \leqslant c\, \|\cdot\|_{B_{(8)}}$, for example if $\sup_{-\infty < \theta \leqslant 0}(Q_n(\theta)/Q(\theta)) < \infty$. In that case, we could consider both problems in the space $B_{(8)}$. The problem becomes more complicated, though, when we want to include the unperturbed problem (1.7). The natural phase space for that problem is $C([-T, 0], \mathbb{R}^n)$. We need to specify a space then in which the norm incorporates the essential properties of both the norms $\|\cdot\|_{B(8)}$ and $\|\cdot\|_{C([-T,0],\mathbb{R}^n)}$. A natural choice would be $\bar{B} := \{\phi : \phi \in B_{(8)}$ and $\phi(\theta)$ is continuous for $\theta \in [-T, 0]$ and $\|\cdot\|_{\bar{B}} < \infty\}$ where

$$\|\phi\|_{\bar{B}} = \sup_{-T \leqslant \theta \leqslant 0} |\phi(\theta)| + \int_{-\infty}^{-T} Q_n(\theta)\, |\phi(\theta)|\, d\theta.$$

Unfortunately, however, we won't be able to obtain the convergence of the nonlinearities in this norm. We will need to enlarge slightly the interval in which the sup is taken. Hence, we will consider the problems (1.7) and $(1.8)_n$ in the space $\bar{B}_v := \{\phi : \phi \in B_{(8)}$ and $\phi(\theta)$ is continuous for $\theta \in [-T-v, 0]$ and $\|\cdot\|_{\bar{B}} < \infty\}$ where

$$\|\phi\|_{\bar{B}} = \sup_{-T-v \leqslant \theta \leqslant 0} |\phi(\theta)| + \int_{-\infty}^{-T} Q_n(\theta)\, |\phi(\theta)|\, d\theta$$

and where v is any positive real number.

In general, if B_λ is the natural phase space associated with the problem $(1.1)_\lambda$ and if all of the delays are infinite, it is natural to try to compare the orbits for different parameter values in the space $\bar{B} := \bigcap_{\lambda \geqslant 0} B_\lambda$. If the limiting problem has finite delay, r, then we define $\bar{B} := \{\phi : \phi \in \bigcap_{\lambda > 0} B_\lambda$ and ϕ is continuous in the inverval $[-r-v, 0]\}$.

If all of the delays, $r(\lambda)$, are finite, we will work in the phase space $\bar{B} = C([-\bar{r}, 0], \mathbb{R}^n)$ where $\bar{r} = \sup_\lambda r(\lambda)$.

We will only consider these situations. To make the writing less cumbersome, we will refer to problems for which functions in $\bar{B}$ take values on the infinite interval $(-\infty, 0]$ as "parameter dependent infinite delay problems" and those for which the functions in $\bar{B}$ are defined on a finite interval as "parameter dependent finite delay problems".

Now that we have chosen a phase space, $\bar{B}$, we must show that we can actually define all of the flows $(1.1)_\lambda$ in $\bar{B}$. Of course, $\bar{B}$ may not even be a Banach space and, if it is, the axioms (A1) through (A5) may not hold in $\bar{B}$. Therefore, we require that $\bar{B}$ is a suitable phase space; that is, $\bar{B}$ satisfies the hypothesis

(PS) $\bar{B}$ is a Banach space with norm $\|\cdot\|_{\bar{B}}$ and axioms (A1) through (A5) hold in $\bar{B}$.

If $\bar{B}$ does satisfy (PS) and all of the delays are infinite, then the flow for each problem $(1.1)_\lambda$ can be defined in $\bar{B}$ in the following way. Fix a value of λ. Suppose that $\bar{\phi}$ is an element in $\bar{B}$. Then $\phi = \bar{\phi}$ is an initial condition in B_λ and the flow through ϕ under $(1.1)_\lambda$, which we denote by $x_t(\cdot\,; \phi, \lambda)$, exists in B_λ for $t \in [0, a_\lambda)$ where $a_\lambda \leqslant \infty$. For each $t \in [0, a_\lambda)$, we can define the function $\bar{x}_t(\cdot\,; \bar{\phi}, \lambda) : (-\infty, 0] \to \mathbb{R}^n$ by $\bar{x}_t(\theta; \bar{\phi}, \lambda) = x_t(\theta; \phi, \lambda)$. We must show that $\bar{x}_t(\cdot\,; \bar{\phi}, \lambda) \in \bar{B}$. If we consider the function $\bar{x}(t) := \bar{x}_t(\cdot\,; \bar{\phi}, \lambda)(0)$, then $\bar{x}$ satisfies the conditions in (A1) since $\bar{x}_0 = \bar{\phi} \in \bar{B}$ and $\bar{x}(t) = x(t; \phi, \lambda)$ is continuous for $t \in [0, a_\lambda)$. Therefore, $\bar{x}_t(\cdot\,; \bar{\phi}, \lambda) = \bar{x}_t(\cdot) \in \bar{B}$. Also, since $\bar{x}(t; \bar{\phi}, \lambda) = x_t(\cdot, \phi, \lambda)(0)$, we have, from the inequalities in axioms (A1) and (A3),

$$\begin{aligned} \|\bar{x}_t\|_{\bar{B}} &\leqslant K \sup_{0 \leqslant s \leqslant t} |\bar{x}(s; \bar{\phi}, \lambda)| + M(t)\, \|x_0\|_{\bar{B}} \\ &\leqslant KH \sup_{0 \leqslant s \leqslant t} \|x_s\|_{B_\lambda} + \bar{M}\, \|x_0\|_{\bar{B}}. \end{aligned} \tag{2.1}$$

We call $\bar{x}_t(\cdot\,; \bar{\phi}, \lambda)$ the flow through $\bar{\phi}$ for $t \in [0, \alpha_\lambda)$ in the phase space $\bar{B}$.

Next, we define what is meant by the flow of $(1.1)_\lambda$ in $\bar{B}$ when the delays are all finite. Again, fix a value of λ. If $\bar{\phi}$ is an element in $\bar{B}$, then $\phi := \bar{\phi}|_{[-r(\lambda), 0]}$ is an initial condition in $C_{r(\lambda)} := C([-r(\lambda), 0], \mathbb{R}^n)$. The flow through ϕ in $C_{r(\lambda)}$ is defined on an interval $[0, \alpha_\lambda)$, $\alpha_\lambda \leqslant \infty$, and we denote it by $x_t(\cdot; \phi, \lambda)$. If we define the function $\bar{x}(t; \bar{\phi}, \lambda) = [x(\cdot; \phi, \lambda)](0)$ for $t \in [-r(\lambda), \alpha_\lambda)$ and $\bar{x}(t; \bar{\phi}, \lambda) = \bar{\phi}(t)$ for $t \in [-\bar{r}, -r(\lambda)]$, then $\bar{x}$ is continuous in $[-\bar{r}, 0]$. Therefore, if we define $\bar{x}_t(\theta; \bar{\phi}, \lambda) := \bar{x}(t + \theta; \bar{\phi}, \lambda)$ for $\theta \in [-\bar{r}, 0]$ and $t \in [-\bar{r}, \alpha_\lambda)$, then $\bar{x}_t(\cdot; \bar{\phi}, \lambda)$ is in $\bar{B}$. We call $\bar{x}_t(\cdot; \bar{\phi}, \lambda)$ the flow through $\bar{\phi}$ for $t \in [0, \alpha_\lambda)$ in the space $\bar{B}$. We will also need to define the backward flow through $\bar{\phi}$ if it exists. Suppose there is a backward orbit $x_t(\cdot; \phi, \lambda)$, $t \leqslant 0$ through ϕ in $C_{r(\lambda)}$. Then, if $\bar{\phi}(\theta) = x(\theta; \phi, \lambda)$ for $\theta \in [-\bar{r}, -r(0)]$, we can define the corresponding backward orbit through $\bar{\phi}$ as follows. Let $\bar{x}(t; \bar{\phi}, \lambda) := [x_t(\cdot; \phi, \lambda)](0)$, $t < 0$. Then $\bar{x}_t(\theta; \bar{\phi}, \lambda) = \bar{x}(t + \theta, \bar{\phi}, \lambda)$. Since $\bar{x}(t; \bar{\phi}, \lambda)$ is continuous, clearly $\bar{x}_t(\cdot; \bar{\phi}, \lambda) \in \bar{B}$ for $t < 0$. If $\bar{\phi}(\theta) \neq x(\theta; \phi, \lambda)$ for all $\theta \in [-\bar{r}, -r(0)]$, then the backward orbit through $\bar{\phi}$ is not defined.

If for $\lambda > 0$ the delays are infinite and the limiting problem has finite delay, r, then we define the flow in $\bar{B}$ for $\lambda > 0$ as we did in the case where all the delays were infinite. For $\lambda = 0$, we follow the argument for the finite delay case. That is, if $\bar{\phi}$ is an element in $\bar{B}$, then $\phi := \bar{\phi}|_{[-r, 0]}$ is an initial condition in $C_r := C([-r, 0], \mathbb{R}^n)$. The flow through ϕ in C_r is defined on an interval $[0, \alpha_\lambda)$, $\alpha_\lambda \leqslant \infty$, and we denote it by $x_t(\cdot; \phi, \lambda)$. Again, we define the function $\bar{x}(t; \bar{\phi}, \lambda) = [x_t(\cdot; \phi, \lambda)](0)$ for $t \in [-r, \alpha_\lambda)$ and $\bar{x}(t; \bar{\phi}, \lambda) = \bar{\phi}(t)$ for $t \in (-\infty, -r]$. Since $\bar{x}(t; \phi, \lambda)$ is continuous for $t \in [-r, \alpha_\lambda)$, it is clear that $\bar{x}$ satisfies the conditions in axiom (A1). Therefore, if we define $\bar{x}_t(\theta; \bar{\phi}, \lambda) := \bar{x}(t + \theta; \bar{\phi}, \lambda)$ for $\theta \in (-\infty, 0]$ and $t \in [0, \alpha_\lambda)$ then $\bar{x}_t(\cdot; \bar{\phi}, \lambda)$ is in $\bar{B}$. We call $\bar{x}_t(\cdot; \bar{\phi}, \lambda)$ the flow through $\bar{\phi}$ for $t \in [0, \alpha_\lambda)$ in the space $\bar{B}$. We will also need to define the backward flow through $\bar{\phi}$ if it exists. Suppose there is a backward orbit $x_t(\cdot; \phi, \lambda)$, $t \leqslant 0$ through ϕ in C_r. Then, if $\bar{\phi}(\theta) = x(\theta; \phi, \lambda)$ for $\theta \in (-\infty, -r]$, we can define the corresponding backward orbit through $\bar{\phi}$ as follows. Let $\bar{x}(t; \bar{\phi}, \lambda) := [x_t(\cdot; \phi, \lambda)](0)$, $t < 0$. Then $\bar{x}_t(\theta; \bar{\phi}, \lambda) = \bar{x}(t + \theta; \bar{\phi}, \lambda)$. Since $\bar{x}(t; \bar{\phi}, \lambda)$ is continuous, clearly $\bar{x}_t(\cdot; \bar{\phi}, \lambda) \in \bar{B}$ for $t < 0$. Again, if $\bar{\phi}(\theta) \neq x(\theta; \phi, \lambda)$ for all $\theta \in (-\infty, -r]$, then the backward orbit through $\bar{\phi}$ is undefined.

3. The Local Theory

A local theory for functional differential equations in a Banach space B satisfying (PS) has been developed by Hale and Kato [1978] and Hino, Murakami and Naito [1991], but in this paper we will need to use local results which are uniform in the parameter and in bounded sets. Here we will develop such a theory for a sequence of retarded functional differential equations

$$\dot{x}(t) = f_\lambda(x_t) \tag{3.1$_\lambda$}$$

in a Banach space B satisfying (PS); that is, $f_\lambda: B \to \mathbb{R}^n$ for each λ. If f_λ is Lipschitz on bounded sets of B, then we can use a contraction mapping argument to show local existence, uniqueness and continuous dependence of initial conditions for each λ. In fact, we will have that $T_\lambda(t)$ is continuous with respect to initial conditions uniformly in bounded sets of B. If, in addition, for each bounded set $\Gamma \subset B$, the Lipschitz constants $l_\lambda(\Gamma)$ of f_λ on Γ satisfy $\sup_\lambda l_\lambda(\Gamma)$ is finite, then we will have uniformity in λ. If $f_\lambda \to f_0$ as $\lambda \to 0$ uniformly in a bounded set Ω, then, for some $\alpha > 0$ sufficiently small, we will have convergence of the solutions $x_\alpha(\cdot\,; \phi, \lambda) \to x_\alpha(\cdot\,; \phi, 0)$ uniformly for $\phi \in \Omega$. Since our equations are autonomous, convergence of $x_\alpha(\cdot\,; \phi, \lambda)$ will be a major tool in the proof of upper semicontinuity. The ideas in the theory below are from Hale [1977]. We begin with a Lemma.

LEMMA 3.1. *Let W be a bounded set in B and for each $\phi \in W$, define $\hat{\phi}$ as follows: $\hat{\phi}(t) = \phi(t)$ for $t \in [-\infty, 0]$* and *$\hat{\phi}(t) = \phi(0)$ for $t > 0$. Let $\{f_\lambda\}$ be sequence of functionals from B to $\mathbb{R}^n$ which are Lipschitz continuous on bounded sets. For any positive real numbers α and ξ we can define the sets $A(\alpha, \xi) = \{ y \in C((-\infty, \alpha], \mathbb{R}^n) : y_0 = 0$ and $|y(t)| < \xi$ for all $t \in [0, \alpha]\}$, $U(\alpha, \xi) = \{ p \in B : p = \hat{\phi}_t + y_t$ where $\phi \in W$ and $y \in A(\alpha, \xi)$, $t \in [0, \alpha]\}$ and $V(\alpha, \xi) = \{\phi + \psi : \phi \in W \cup U(\alpha, \xi), \psi \in B$ and $\|\psi\|_B \leqslant \xi\}$. $A(\alpha, \xi)$ will be a set containing orbits of the equation*

$$\dot{y}(t) = f_\lambda(\hat{\phi}_t + y_t)$$

$$y_0 = 0$$

and $U(\alpha, \xi)$ will contain the orbits $x_t = \hat{\phi}_t + y_t$. There exist positive constants $M(\xi)$ and $\hat{\lambda}$ so that $|f_\lambda(\phi)| < M(\xi)$ for all $\phi \in V(\alpha, \xi)$ and for all $\lambda \leqslant \hat{\lambda}$. If $l_\lambda(V)$ is the Lipschitz constant of f_λ on V, then we can define $l(V) = \sup_{\lambda \leqslant \hat{\lambda}} l_\lambda(V)$. It is possible to choose $M(\xi)$, α and ξ so that $M(\xi)\alpha < \xi$ and $l(V)\alpha < 1$.

Proof. The proof is trivial. Let K_W be a constant such that, if $\phi \in W$, then $\|\phi\| \leqslant K_W$. Let $v \in V$ and $v = \phi + \psi$. If $\phi \in W$, then $\|v\|_B \leqslant K_W + \xi$, and if $\phi \in U(\alpha, \xi)$, then $\|\phi\|_B \leqslant K_W + \|y_t\|_B$, where $y \in A(\alpha, \xi)$. But from the inequality in (A1), $\|y_t\|_B \leqslant K \sup_{0 \leqslant s \leqslant t} |y(s)|$. Therefore,

$$\|v\|_B \leqslant K_W + \|y_t\|_B + \xi \leqslant K_W + \sup_{0 \leqslant s \leqslant t} |y(s)| + \xi \leqslant K_W + 2\xi$$

and so V is a bounded set of B and since f_λ maps bounded sets to bounded sets there exists a positive constant $M(\xi, \lambda)$ so that $|f_\lambda(\phi)| < M(\xi, \lambda)$. Of course, we can pick a $\hat{\lambda}$ so that $M(\xi) = \sup_{\lambda \leqslant \hat{\lambda}} M(\xi, \lambda)$ is finite. Since the bound on V depends only on ξ and K_W, we can always pick α to satisfy the inequalities.

We can use this Lemma to prove local existence and uniqueness of solutions. The interval of existence will be uniform for bounded sets of initial conditions and uniform in the parameter λ.

THEOREM 3.2. *For any bounded set $W \subset B$ of initial conditions and any sequence of functionals $\{f_\lambda\}$, there is a time, $\alpha(W)$, and a $\hat{\lambda}(W)$ such that for each $\phi \in W$ and for each $\lambda \leqslant \hat{\lambda}(W)$, there is a unique solution of $(3.1)_\lambda$ through ϕ which exists on $[0, \alpha]$.*

Proof. Pick α, ξ, $M(\xi)$ and $\hat{\lambda}(W)$ as in Lemma 3.1. We remark that α depends only on ξ and the set W. Define the map G: $W \times \{f_\lambda : \lambda \leqslant \hat{\lambda}\} \times A(\alpha, \xi) \to C([-\infty, \alpha], \mathbb{R}^n)$ as follows;

$$G(\phi, f_\lambda, y) = 0 \qquad t \in (-\infty, 0]$$

$$G(\phi, f_\lambda, y)(t) = \int_0^t f(\hat{\phi}_s + y_s)\, ds \qquad t \in [0, \alpha].$$

If y is a fixed point of $G(\phi, f_\lambda, \cdot)$, then $x_t = \hat{\phi}_t + y_t$ is a solution of $(3.1)_\lambda$ on $[0, \alpha]$, so we will use the contraction mapping principle to show existence and uniqueness of a fixed point in $A(\alpha, \xi)$ for each $\phi \in W$ and for each f_λ.

It is clear that $G(\phi, f_\lambda, y)$ is continuous in t since

$$|G(\phi, f_\lambda, y)(t_1) - G(\phi, f_\lambda, y)(t_2)| \leqslant \int_{t_2}^{t_1} |f_\lambda(\hat{\phi}_s + y_s)|\, ds \leqslant M(\xi)\, |t_1 - t_2|.$$

Also $G(\phi, f_\lambda, y)_0 = 0$ and

$$|G(\phi, f_\lambda, y)(t)| \leqslant \int_0^t |f_\lambda(\hat{\phi}_s + y_s)|\, ds \leqslant M(\xi)\alpha \leqslant \xi,$$

so $G(\phi, f_\lambda, y) \in A(\alpha, \xi)$ and, in fact, the image of G is compact in $A(\alpha, \xi)$. For fixed ϕ and f_λ, $\lambda \leqslant \hat{\lambda}$, $G(\phi, f_\lambda, y)$ is a contraction on $A(\alpha, \xi)$; that is for $y, \bar{y} \in A(\alpha, \xi)$,

$$\begin{aligned}
&|G(\phi, f_y, y)(t) - G(\phi, f_\lambda, \bar{y})(t)| \\
&\quad \leqslant \int_0^t |f(\hat{\phi}_s + y_s) - f(\hat{\phi}_s + \bar{y}_s)|\, ds \\
&\quad \leqslant \int_0^t l(V)\, \|y_s - \bar{y}_s\|_B\, ds \\
&\quad \leqslant l(V)\alpha \sup_{0 \leqslant s \leqslant \alpha} \|y_s - \bar{y}_s\|_B
\end{aligned}$$

$$\leqslant l(V)\alpha \sup_{0\leqslant s\leqslant\alpha} \left[\sup_{0\leqslant\theta\leqslant s} |y(\theta)-\bar{y}(\theta)| + M_1(s)\, \|y_0-\bar{y}_0\|_B \right]$$

$$\leqslant l(V)\alpha \sup_{-\infty\leqslant\sigma\leqslant\alpha} |y(\sigma)-\bar{y}(\sigma)|$$

$$= l(V)\alpha\, \|y-\bar{y}\|_{C([-\infty,\phi],\mathbb{R}^n)}$$

so $\|G(\phi,f_\lambda,y)-G(\phi,f_\lambda,\bar{y})\|_{C([-\infty,\alpha],\mathbb{R}^n)} \leqslant l(V)\alpha\, \|y-\bar{y}\|_{C([-\infty,\alpha],\mathbb{R}^n)}$ where $l(V)\alpha<1$. Therefore, we can apply the contraction mapping principle to show that there is a unique fixed point $y=g(\phi,f_\lambda)\in A(\alpha,\xi)$ of $G(\phi,f_\lambda,\cdot)$. We can do this for every $\phi\in W$ since α depends on W, not ϕ. Hence the Theorem is proved.

Next we prove, by using the standard estimates, that we have continuity with respect to initial conditions uniformly in bounded sets and uniformly for the parameter. The same type of estimates can be used to prove convergence of solutions when $f_\lambda\to f_0$.

THEOREM 3.3. *If $y=g(\phi,f_\lambda)$ is the fixed point of the map $G(\phi,f_\lambda,\cdot)$ defined above and if $\alpha l(V)K<1$ where K is the constant in the inequality in* (A1), *then g as an element of $C((-\infty,\alpha],\mathbb{R}^n)$ is continuous in ϕ, uniformly in bounded sets. Also, for $t\in[0,\alpha]$, $g_t(\phi,f_\lambda)$ as an element in B is continuous in ϕ uniformly in bounded sets of B and uniformly in t. If in addition, $f_\lambda\to f_0$ uniformly in bounded sets of B, then, as a sequence in $C((-\infty,\alpha],\mathbb{R}^n)$, $g(\phi,f_\lambda)\to g(\phi,f_0)$ uniformly in bounded sets of initial conditions and, as a sequence in B, $g_t(\phi,f_\lambda)\to g_t(\phi,f_0)$ uniformly in bounded sets of B and uniformly for $t\in[0,\alpha]$.*

Proof. For fixed λ and for ϕ, ψ in a bounded set W, define $\bar{y}=g(\phi,f_\lambda)$ and $y=g(\psi,f_\lambda)$. Then

$$\sup_{-\infty\leqslant s\leqslant\alpha} |g(\phi,f_\lambda)(s)-g(\psi,f_\lambda)(s)|$$

$$= \sup_{-\infty\leqslant s\leqslant\alpha} |G(\phi,f_\lambda,\bar{y})-G(\psi,f_\lambda,y)|$$

$$\leqslant \sup_{-\infty\leqslant s\leqslant\alpha} |G(\phi,f_\lambda,\bar{y})-G(\phi,f_\lambda,y)|$$

$$+ \sup_{-\infty\leqslant s\leqslant\alpha} |G(\phi,f_\lambda,y)-G(\psi,f_\lambda,y)|$$

$$\leqslant \int_0^\alpha |f_\lambda(\hat{\phi}_s+\bar{y}_s)-f_\lambda(\hat{\phi}_s+y_s)|\,ds + \int_0^\alpha |f_\lambda(\hat{\phi}_s+y_s)-f_\lambda(\hat{\psi}_s+y_s)|\,ds$$

$$\leqslant l(V)\int_0^\alpha \|\bar{y}_s - y_s\|_B\,ds + l(V)\int_0^\alpha \|\hat{\phi}_s - \hat{\psi}_s\|_B\,ds$$

$$\leqslant \alpha l(V) \sup_{0\leqslant s\leqslant\alpha} \|\bar{y}_s - y_s\|_B + \alpha l(V) \sup_{0\leqslant s\leqslant\alpha} \|\hat{\phi}_s - \hat{\psi}_s\|_B$$

$$\leqslant \alpha l(V) K\Bigg[\sup_{0\leqslant\theta\leqslant\alpha}\ \sup_{0\leqslant s\leqslant\alpha} |\bar{y}(\theta) - y(\theta)|$$

$$+ \sup_{0\leqslant s\leqslant\alpha}\left[\sup_{0\leqslant\theta\leqslant s} |\hat{\phi}(\theta) - \hat{\psi}(\theta)| + \frac{M(s)}{K}\|\hat{\phi}_0 - \hat{\psi}_0\|_B\right]\Bigg]$$

$$\leqslant \alpha l(V)\Bigg[K \sup_{-\infty\leqslant s\leqslant\alpha} |g(\phi, f_\lambda)(s) - g(\psi, f_\lambda)(s)|$$

$$+ K|\phi(0) - \psi(0)| + \bar{M}\|\phi - \psi\|_B\Bigg]$$

Therefore

$$\sup_{-\infty\leqslant s\leqslant\alpha} |g(\phi, f_\lambda)(s) - g(\psi, f_\lambda)(s)| \leqslant \frac{\alpha l(V)(K + \bar{M}])}{1 - \alpha l(V)K}\|\phi - \psi\|_B$$

and so $g(\cdot, f_\lambda)$ is continuous in the sup norm provided $\alpha l(V)K < 1$. Similarly,

$$\sup_{-\infty\leqslant s\leqslant\alpha} |g(\phi, f_\lambda)(s) - g(\phi, f_0)(s)|$$

$$\leqslant \frac{1}{1 - \alpha l(V)K}\int_0^\alpha |f_\lambda(\hat{\phi}_s + g(\phi, f_0)_s) - f_0(\hat{\phi}_s + g(\phi, f_0)_s)|\,ds$$

and so if $f_\lambda \to f_0$ uniformly on bounded sets then $g(\phi, f_\lambda) \to g(\phi, f_0)$ uniformly on bounded sets in the $C((-\infty, \alpha], \mathbb{R}^n)$-norm. Since g is 0 for $s < 0$, we can use the inequality in (A1) to get, for $t \in [0, \alpha]$,

$$\|g_t(\phi, f_\lambda) - g_t(\psi, f_\lambda)\|_B \leqslant K \sup_{0\leqslant s\leqslant\alpha} |g(\phi, f_\lambda)(s) - g(\psi, f_\lambda)(s)|$$

$$\leqslant K\frac{\alpha l(V)(K + \bar{M})}{1 - \alpha l(V)K}\|\phi - \psi\|_B$$

and

$$\|g_t(\phi, f_\lambda) - g_t(\phi, f_0)\|_B$$

$$\leqslant \frac{K}{1 - \alpha l(V)K}\int_0^\alpha |f_\lambda(\hat{\phi}_s + g(\phi, f_0)_s) - f_0(\hat{\phi}_s + g(\phi, f_0)_s)|\,ds.$$

And so the Theorem is proved.

In fact, if $T_\lambda(t)$ is a bounded operator for each $t>0$, then it is locally Lipschitz (in time) on bounded sets. That is, if $\Gamma \subset B$ is bounded, then there exists a function $l(\Gamma, t)$, continuous in t, such that for each $\phi, \psi \in B$, $\|T_\lambda(t)\phi - T_\lambda(t)\psi\|_B \leqslant l(\Gamma, t)\ \|\phi-\psi\|_B$

Now suppose $f_\lambda \to f_0$ uniformly in C_r-equicontinuous sets and that $\Omega \subset B$ is C_r-equicontinuous for some $r>0$.If we can prove that the argument $\phi_s + [g(\phi, f_0)]_s$ of f_λ and f_0 is in a C_r-equicontinuous set, then we will be able to prove, in the same manner as Theorem 3.3

THEOREM 3.4. *Suppose that $f_\lambda \to f_0$ uniformly on C_r-equicontinuous sets and that Ω is a C_r-equicontinuous set. Then, for each ε, there exists $\hat{\lambda}$ such that for all $\lambda \leqslant \hat{\lambda}$ and for all $\phi \in \Omega$ and for all $t \in [0, \alpha]$, $\|g_t(\phi, f_\lambda) - g_t(\phi, f_0)\|_B \leqslant \varepsilon$.*

Proof. Since the image of G is compact in $A(\alpha, \xi)$ and since the set $\hat{\Omega} := \{\hat{\phi}_s : \phi \in \Omega, s \in [0, \alpha]\}$ is C_r-equicontinuous, the argument $\hat{\phi}_s + [g(\phi, f_0)]_s$ is in a C_r-equicontinuous set and so the Theorem is proved.

4. THE RESULTS

In the remainder of the paper, we will often use the semigroup notation for solutions of $(1.1)_\lambda$. By $T_\lambda(t)$ we mean the solution semigroup associated with $(1.1)_\lambda$ in the space B_λ and by $\bar{T}_\lambda(t)$, we mean the solution semigroup associated with $(1.1)_\lambda$ in the space $\bar{B}$.

Suppose the problem $(1.1)_0$ admits a global attractor, $\mathscr{A}_0 \in B_0$. As a first step, we must find a compact set $\bar{\mathscr{A}}_0 \in \bar{B}$ which is a global attractor for the flow $(1.1)_0$ in $\bar{B}$, as defined in Section 2. We have the following Theorem

THEOREM 4.1. *Suppose there is a global attractor for the problem $(1.1)_0$ in B_0. If the space $\bar{B}$ is defined as in section 2 and $\bar{B}$ satisfies the condition* (PS), *then there is a global attractor for the flow $(1.1)_0$ in the space $\bar{B}$.*

Now we can state the results about existence and upper semicontinuity of attractors in the space $\bar{B}$. Suppose the problem $(1.1)_0$ admits a global attractor $\bar{\mathscr{A}}_0 \in \bar{B}$. Then, intuitively, if the perturbation in $(1.1)_\lambda$ is "small", there should be a global attractor for $(1.1)_\lambda$, $\bar{\mathscr{A}}_\lambda$, which is "close" to $\bar{\mathscr{A}}_0$. It would be reasonable to expect that the hypothesis "$f_\lambda \to f_0$ as $\lambda \to 0$ uniformly on bounded sets of $\bar{B}$" might give us existence of global attractors for λ small enough and upper semicontinuity of the attractors at $\lambda = 0$. This is not quite the case, however. Instead we get local attractors which are upper semicontinuous and satisfy

(A) For each bounded set $\bar{\Gamma} \subset \bar{B}$ and for each $\varepsilon > 0$, there is a $\hat{\lambda}(\bar{\Gamma}, \varepsilon)$ such that for all $\lambda \leqslant \hat{\lambda}$, the ω-limit set $\omega_\lambda(\bar{\Gamma})$ enters an ε-neighborhood of $\bar{\mathscr{A}}_0$.

Since for any bounded set $\bar{\Gamma}$ which contains $\bar{\mathscr{A}}_0$, $\omega(\bar{\Gamma}) = \bar{\mathscr{A}}_0$, this is equivalent to upper semicontinuity of ω-limit sets of sets containing $\bar{\mathscr{A}}_0$. This may be sufficient for many applications.

We need stronger uniformity in λ to obtain global attractors. We will consider two different hypotheses which will give us such uniformity. The first is a strong hypothesis about the convergence of the nonlinearities.

(U) $f_\lambda \to f_0$ uniformly in $\bar{B}$ and each f_λ is globally Lipschitz in $\bar{B}$ with Lipschitz constant l_λ and $\sup_\lambda l_\lambda < \infty$

The second is an a priori assumption about bounded dissipativeness of the problems $(1.1)_\lambda$. We will refer to this property as *uniform bounded dissipativeness.*

(UBD) There is a ball, $\bar{b}(R) \supset \bar{\mathscr{A}}_0$, of radius R which is independent of λ, into which the orbit $\bar{T}_\lambda(t)\bar{\Gamma}$ eventually enters and remains for every bounded set $\bar{\Gamma}$.

As indicated in section 3, we will also need a hypothesis about Lipschitz continuity of the nonlinearities f_λ.

(L) f_λ is Lipschitz on bounded sets of $\bar{B}$ and if $\bar{\Gamma}$ is a bounded set and $l_\lambda(\bar{\Gamma})$ is the Lipschitz constant of f_λ on $\bar{\Gamma}$ then $l(\bar{\Gamma}) := \sup_\lambda l_\lambda(\bar{\Gamma})$ is finite.

We then have the following Theorem

THEOREM 4.2. *Suppose there is a global attractor $\bar{\mathscr{A}}_0 \subset \bar{B}$ for the flow $(1.1)_0$ in $\bar{B}$. If the functions $\{f_\lambda\}$ satisfy the Lipschitz condition* (L) *and if $f_\lambda \to f_0$ uniformly in bounded sets of $\bar{B}$ then, for λ small, there is a local attractor $\bar{\mathscr{A}}_\lambda \subset \bar{B}$ for the problem $(1.1)_\lambda$ and the family $\{\bar{\mathscr{A}}_\lambda\}$ is upper semicontinuous at $\lambda = 0$. Furthermore, the attractors $\{\bar{\mathscr{A}}_\lambda\}$ have the property* (A). *If, in addition, either condition* (U) *or condition* (UBD) *can be verified, then the attractors are global.*

In the applications, when the functions $d\eta_{j\lambda}$ have discontinuities, the hypothesis $f_\lambda \to f_0$ as $\lambda \to 0$ uniformly on bounded sets of $\bar{B}$ is generally not satisfied. We may have, for each $\phi \in \bar{B}$, $f_\lambda(\phi) \to f_0(\phi)$ as $\lambda \to 0$, but this is not good enough to get bounded dissipation. In many applications, we do have, however, that $f_\lambda \to f_0$ uniformly on C_r-equicontinuous sets for some $r > 0$. If the solutions of $(1.1)_\lambda$ all exist and remain bounded, uniformly in λ, up to time r, then we can consider the set $F(r, \bar{\Gamma}) := \bigcup_\lambda \bar{T}_\lambda(r)\bar{\Gamma}$, for any bounded set $\bar{\Gamma}$. F is also bounded, so it must be attracted by the set $\bar{\mathscr{A}}_0$ under the flow $(1.1)_0$. If $F(r, \bar{\Gamma})$ is also C_r-equicontinuous and if $f_\lambda \to f_0$ uniformly on C_r-equicontinuous sets, then, for λ small enough, $\bar{T}_\lambda(t)\, F(r, \bar{\Gamma})$ must enter a neighborhood of $\bar{\mathscr{A}}_0$. With this we will obtain a family of local attractors, $\{\bar{\mathscr{A}}_\lambda\}$, which are upper semicontinuous at $\lambda = 0$

and which satsify property (A) above. Therefore, we consider the following three hypotheses.

(c1) For each λ and for each $\phi \in B_\lambda$, the solution, $T_\lambda(t)\phi$, exists and remains bounded, uniformly in λ, for $t \leqslant r$.

(c2) For each bounded set $\bar{\Gamma} \subset \bar{B}$, the set $F(r, \bar{\Gamma}) := \bigcup_\lambda \bar{T}_\lambda(r)\bar{\Gamma}$ is C_r-equicontinuous.

(c3) $f_\lambda \to f_0$ uniformly on C_r-equicontinuous sets

We then have the following theorem.

THEOREM 4.3. *Suppose there is a global attractor* $\bar{\mathscr{A}}_0 \subset \bar{B}$ *for the flow* $(1.1)_0$ *in the space* $\bar{B}$. *If the functions* $\{f_\lambda\}$ *satisfy the Lipschitz condition* (L) *and if the hypotheses* (c1) *through* (c3) *hold, then, for* λ *small, there is a local attractor* $\bar{\mathscr{A}}_\lambda \subset \bar{B}$ *for the problem* $(1.1)_\lambda$ *and the family* $\{\bar{\mathscr{A}}_\lambda\}$ *is upper semi-continuous at* $\lambda = 0$. *Furthermore, the attractors* $\{\bar{\mathscr{A}}_\lambda\}$ *have the property* (A). *If, in addition, condition* (UBD) *can be verified, then the attractors are global.*

5. PROOF OF THEOREM 4.1

First we prove the Theorem for parameter dependent finite delay problems. We begin by constructing the set $\bar{\mathscr{A}}_0$.

If $\bar{r} := \sup_\lambda r(\lambda)$, then $\bar{B}$ will be the space $C_{\bar{r}} := C([-\bar{r}, 0], \mathbb{R}^n)$. B_0 will be the space $C_{r(0)} := C([-r(0), 0], \mathbb{R}^n)$. Let y be an element in $\mathscr{A}_0$ and let $x_t(y)$ be the solution of $(1.1)_0$ through y. Define $x(t; y) := [x_t(y)](0)$. Since $\mathscr{A}_0$ is invariant, for each $y \in \mathscr{A}_0$, the forward and backward orbits through y exist and are in $\mathscr{A}_0$; that is, $x_t(y) \in \mathscr{A}_0$ for all $t \in (-\infty, \infty)$. Therefore, we can consider the set $\tilde{\mathscr{A}} \subset C((-\infty, \infty), \mathbb{R}^n)$ which is made up of orbits of initial conditions in $\mathscr{A}_0$; that is $\tilde{\mathscr{A}} := \{\tilde{y} \in C((-\infty, \infty), \mathbb{R}^n): \tilde{y}(t) = x(t; y), t \in (-\infty, \infty), y \in \mathscr{A}_0\}$. We can recover $\mathscr{A}_0$ from $\tilde{\mathscr{A}}$ simply by taking the restriction to the interval $[-r(0), 0]$ of elements in $\tilde{\mathscr{A}}$. For the set $\bar{\mathscr{A}}_0$ then, we will take the restriction of elements in $\tilde{\mathscr{A}}$ to the interval $[-\bar{r}, 0]$; that is $\bar{\mathscr{A}}_0 := \{\bar{y} \in C([-\bar{r}, 0], \mathbb{R}^n): \bar{y}(\theta) = \tilde{y}(\theta)$ for $\theta \in [-\bar{r}, 0]$, $\tilde{y} \in \tilde{\mathscr{A}}\}$. From here on, we will denote elements of $\bar{\mathscr{A}}_0$ as $\bar{y}$ and the corresponding function in $\mathscr{A}_0$ as y. Alternatively, we can think of an element in $\bar{\mathscr{A}}_0$ as an element in $\mathscr{A}_0$ extended to the interval $[-\bar{r}, 0]$ by its backward flow for time $\bar{r} - r(0)$; that is, $\bar{y}(\theta) = x(\theta; y)$ for $\theta \in [-\bar{r}, 0]$. $\mathscr{A}_0$ can be recovered by taking the restriction to the interval $[-r(0), 0]$ of each $\bar{y} \in \bar{\mathscr{A}}_0$.

We can prove that $\bar{\mathscr{A}}_0$ is an attractor for the flow $(1.1)_0$ considered in $C_{\bar{r}}$. First we prove

LEMMA 5.1. *$\bar{\mathscr{A}}_0$ is compact in $C_{\bar{r}}$.*

Proof. If we can show that $\bar{\mathscr{A}}_0$ is equibounded and equicontinuous, then we can apply the Arzela–Ascoli Theorem to prove the Lemma.

Since $\mathscr{A}_0$ is compact, there is a positive constant K_0, independent of y, such that if $y \in \mathscr{A}_0$ then $\|y\|_{C_{r(0)}} < K_0$; that is $\mathscr{A}_0$ is equibounded in $C_{r(0)}$.

Since $\mathscr{A}_0$ is invariant, $T_0(t)\,\mathscr{A}_0 = \mathscr{A}_0$ for all t. Since f_0 is Lipschitz on bounded sets of $C_{r(0)}$, f_0 maps bounded sets to bounded sets; therefore, there exists an $M_0 > 0$ such that for each $y \in \mathscr{A}_0$, $|f_0(y)| \leqslant M_0$. Then for $s \in [-r(0), 0]$, $|\dot{y}(s)| = |f_0(y_s)| = |f_0(T_0(s)\,y)| \leqslant M_0$ and so the set $\mathscr{A}_0$ is equicontinuous in $C_{r(0)}$.

Now suppose that $\bar{y} \in \bar{\mathscr{A}}_0$. The restriction of $\bar{y}$ to any interval of length $r(0)$ is contained in $\mathscr{A}_0$, provided we translate it to the interval $[-r(0), 0]$. Indeed, if we restrict $\bar{y}$ to the interval $[-a-r(0), -a]$ where $0 < a < \bar{r} - r(0)$, then $\bar{y}|_{[-a-r(0), -a]}(\theta - a) = x_{-a}(\cdot, \bar{y}, 0)|_{[-r(0), 0]}(\theta)$. We will show below that $\bar{\mathscr{A}}_0$ is invariant, so $x_{-a}(\cdot, \bar{y}, 0) \in \bar{\mathscr{A}}_0$. Therefore, by construction, $x_{-a}(\cdot, \bar{y}, 0)|_{[-r(0), 0]} \in \mathscr{A}_0$. From this it is clear that $\bar{\mathscr{A}}_0$ is also bounded by K_0. For proof, we select the following covering of the interval $[-\bar{r}, 0]$: $\mathscr{C} = \{[-\bar{r}, r - \bar{r}],\ [r(0) - \bar{r},\ 2r(0) - \bar{r}],\ \ldots,\ [(k-1)\,r(0) - \bar{r},\ kr(0) - \bar{r}],\ [-r(0), 0]\}$ where k is the largest integer so that $kr(0) - \bar{r} < 0$. Then if $\bar{y} \in \bar{\mathscr{A}}_0$,

$$\|\bar{y}\|_{C_{\bar{r}}} = \sup_{-\bar{r} \leqslant \theta \leqslant 0} |\bar{y}(\theta)| = \max_{I \in \mathscr{C}} \sup_{\theta \in I} |\bar{y}(\theta)| = \max_{I \in \mathscr{C}} \|\bar{y}|_I\|_{C_{r(0)}} < K_0$$

and so $\bar{\mathscr{A}}_0$ is equibounded. Similarly $\sup_{-\bar{r} \leqslant \theta \leqslant 0} |(d/d\theta)\ \bar{y}(\theta)| \leqslant M_0$, and so $\bar{\mathscr{A}}_0$ is equicontinuous. Hence, by the Arzela–Ascoli theorem, $\bar{\mathscr{A}}_0$ is compact.

LEMMA 5.2. *$\bar{\mathscr{A}}_0$ is invariant.*

Proof. To show that $\bar{\mathscr{A}}_0$ is invariant, fix a $t \in (-\infty, \infty)$ and consider the flow $\bar{x}_t(\cdot\,; \bar{y}, 0)$ through an initial condition $\bar{y} \in \bar{\mathscr{A}}_0$. We must show that there is a $y^* \in \mathscr{A}_0$ that generates $\bar{x}_t$ according to our construction; that is $\bar{x}_t = x(s; y^*, 0)|_{[-\bar{r}, 0]}$. But

$$\bar{x}_t(\theta) = \bar{x}(t + \theta; \bar{y}, 0) = x(t + \theta; y, 0) = x(\theta, x_t(\cdot\,; y, 0), 0), \qquad \theta \in [-\bar{r}, 0]$$

so if we let $y^* := x_t(\cdot\,; y, 0)$, then $y^* \in \mathscr{A}_0$ since $\mathscr{A}_0$ is invariant and $\bar{x}_t(\cdot\,; \bar{y}, 0) = x(s, y^*, 0)|_{[-r, 0]}$, so $\bar{x}_t \in \bar{\mathscr{A}}_0$. In fact, the flow through $\bar{y}$ is just time translation followed by restriction to the interval $[-\bar{r}, 0]$ of the function $\bar{x}(t; \bar{y}, 0)$.

To complete the proof, we must show that $\bar{\mathscr{A}}_0$ attracts bounded sets of $C_{\bar{r}}$ under the flow $(1.1)_0$. We know that $\mathscr{A}_0$ attracts bounded sets in $C_{r(0)}$ and the flows in $C_{\bar{r}}$ and $C_{r(0)}$ through a given initial condition ϕ or $\bar{\phi}$ are in some sense the same. In fact, if we view the flows in $\mathbb{R}^n$ then $x(t; \phi, 0) = \bar{x}(t; \bar{\phi}, 0)$ for all $t \geqslant -r(0)$. The choice of space, $C_{\bar{r}}$ or $C_{r(0)}$, just determines the length of the interval in which we view the flow. Therefore, it seems intuitive that the choice of the space should not affect the asymptotic dynamics. This is indeed the case.

LEMMA 5.3. *$\bar{\mathscr{A}}_0$ attracts bounded sets of $C_{\bar{r}}$ under the flow $(1.1)_0$ in $C_{\bar{r}}$.*

Proof. Let $\bar{\Gamma} \subset C_{\bar{r}}$ be bounded and define $\Gamma = \{\phi|_{[-r(0), 0]} : \phi \in \bar{\Gamma}\}$. Then Γ is bounded in $C_{r(0)}$. Since $\mathscr{A}_0$ attracts bounded sets in $C_{r(0)}$, we have, by definition, that for each ε there exists a $\tau > 0$ such that

$$d_{r(0)}(T_0(t)\,\Gamma, \mathscr{A}_0) = \sup_{\phi \in T_0(t)\Gamma} \inf_{\psi \in \mathscr{A}_0} \{ \sup_{-r(0) \leqslant \theta \leqslant 0} |\phi(\theta) - \psi(\theta)| \} < \varepsilon, \forall t > \tau$$

So we have

$$\begin{aligned}
d_{\bar{r}}(\bar{T}(t)\,\bar{\Gamma}, \bar{\mathscr{A}}_0) &= \sup_{\bar{\phi} \in \bar{\Gamma}} \inf_{\bar{\psi} \in \bar{\mathscr{A}}_0} \{ \sup_{-\bar{r} \leqslant \theta \leqslant 0} |[\bar{T}_0(t+\theta)\bar{\phi}](0) - \bar{\psi}(0)| \} \\
&= \sup_{\bar{\phi} \in \bar{\Gamma}} \inf_{\bar{\psi} \in \bar{\mathscr{A}}_0} \{ \sup_{-\bar{r} \leqslant \theta \leqslant 0} |[\bar{T}_0(\theta+\bar{r})\,\bar{T}_0(t-\bar{r})\bar{\phi}](0) \\
&\quad - [\bar{T}_0(\theta+\bar{r})\,\bar{T}_0(t-\bar{r})\bar{\psi}](0)| \} \\
&= \sup_{\phi \in \Gamma} \inf_{\psi \in \mathscr{A}_0} \{ \sup_{-\bar{r} \leqslant \theta \leqslant 0} |[T_0(\theta+\bar{r})\,T_0(t-\bar{r})\phi](0) \\
&\quad - [T_0(\theta+\bar{r})\,T_0(t-\bar{r})\psi](0)| \} \\
&\leqslant \sup_{\phi \in \Gamma} \inf_{\psi \in \mathscr{A}_0} \{ \sup_{-\bar{r} \leqslant \theta \leqslant 0} l_{T_0(\theta+\bar{r})}(\gamma(\phi) \cup \mathscr{A}_0) \\
&\quad \cdot \|T_0(t-\bar{r})\phi - T_0(t-\bar{r})\psi\| \} \\
&\leqslant \sup_{\phi \in \Gamma} \inf_{\psi \in \mathscr{A}_0} \{ \sup_{0 \leqslant \theta \leqslant \bar{r}} l_{T_0(\theta)}(\gamma(\phi) \cup \mathscr{A}_0) \\
&\quad \cdot \|T_0(t-\bar{r})\phi - T_0(t-\bar{r})\psi\| \} \\
&\leqslant \varepsilon \cdot \sup_{0 \leqslant \theta \leqslant \bar{r}} l_{T_0(\theta)}(\gamma(\phi) \cup \mathscr{A}_0) \qquad \text{for all} \quad t > \tau + \bar{r}
\end{aligned}$$

where $l_{T_0(\theta)}(\gamma(\phi) \cup \mathscr{A}_0)$ is the Lipschitz constant of $T_0(t)$ on $\gamma(\phi) \cup \mathscr{A}_0$ for $t \in [0, \bar{r}]$. Hence $\bar{\mathscr{A}}_0$ attracts bounded sets in $C_{\bar{r}}$ and the Lemma is proved.

Therefore, the set $\bar{\mathscr{A}}_0$ is indeed a global attractor for the flow $(1.1)_0$ in $\bar{B}$ and we have proven Theorem 4.1 for the case when $\bar{r} < \infty$.

Now suppose $\bar{r} = \infty$ but the limiting problem $(1.1)_0$ has finite delay, r. In that case, B_0 will be the space $C([-r, 0], \mathbb{R}^n)$ but $\bar{B}$ will contain functions defined on the whole interval $(-\infty, 0]$. We construct $\bar{\mathscr{A}}_0$ in the same way as above. Suppose $y \in \mathscr{A}_0$. The forward and backward orbits of y exist and are subsets of $\mathscr{A}_0$. As before, define $x(t; y, 0) = [x_t(\cdot, y, 0)](0)$ for $t \in (-\infty, \infty)$. Clearly, $x(t; y, 0) \in BC((-\infty, \infty), \mathbb{R}^n)$, hence, if we define $\bar{y} = x(t; y, 0)|_{(-\infty, 0]}$, then, by axiom (A4), $\bar{y} \in \bar{B}$. We then define $\bar{\mathscr{A}}_0 = \{\bar{y}: y \in \mathscr{A}_0\}$. If the limiting problem has infinite delay, then both B_0 and $\bar{B}$ are spaces containing functions defined on $(-\infty, 0]$. Since $\mathscr{A}_0 \subset BC((-\infty, 0], \mathbb{R}^n)$, by axiom (A4), $\mathscr{A}_0 \subset \bar{B}$. Therefore, we just define $\bar{\mathscr{A}}_0 = \mathscr{A}_0$. We can prove that for these parameter dependent infinite delay problems $\bar{\mathscr{A}}_0$ is an attractor for the flow $(1.1)_0$ in $\bar{B}$.

Again we begin with the Lemma

LEMMA 5.4. *$\bar{\mathscr{A}}_0$ is compact*

Proof. First we show that $\bar{\mathscr{A}}_0$ is equibounded and equicontinuous in $C((-\infty, 0], \mathbb{R}^n)$.

The proof that $\mathscr{A}_0$ is equibounded is basically the same as above. Since $\mathscr{A}_0$ is compact, there exists a positive constant, K_0, such that if $y \in \mathscr{A}_0$ then $\|y\|_{B_0} \leqslant K_0$. Since $\mathscr{A}_0$ is invariant, for any $y \in \mathscr{A}_0$, $T_0(t)\, y \in \mathscr{A}_0$ for all $t \in (-\infty, \infty)$; in fact $T_0(t)\, \mathscr{A}_0 = \mathscr{A}_0$. Then, for $s \in [-r, 0]$, we have

$$|y(s)| = |[T_0(s)\, y](0)| \leqslant H\, \|T_0(s)\, y\|_{B_0} \leqslant HK_0$$

and so $\mathscr{A}_0$ is equibounded in $C([-r, 0], \mathbb{R}^n)$.

The proof that $\mathscr{A}_0$ is equicontinuous is exactly the same as the finite delay case.

If the limiting equation has infinite delay, we are done. Otherwise the proof that $\bar{\mathscr{A}}_0$ is equibounded in $C((-\infty, 0], \mathbb{R}^n)$ is the same as in the previous case except now we choose the infinite covering $\mathscr{C} = \{[-r(0), 0], [-2r(0), -r(0)], ...\}$. Then

$$\|\bar{y}\|_{C((-\infty, 0], \mathbb{R}^n)} = \sup_{-\infty < \theta \leqslant 0} |\bar{y}(\theta)| = \sup_{I \in \mathscr{C}} \sup_{\theta \in I} |\bar{y}(\theta)| = \sup_{I \in \mathscr{C}} \|\bar{y}|_I\|_{C_{r(0)}} \leqslant HK_0$$

Similarly, $\sup_{-\infty < s \leqslant 0} |\dot{\bar{y}}(s)| \leqslant M_0$, and so $\bar{\mathscr{A}}_0$ is equibounded and equicontinuous in $C((-\infty, 0], \mathbb{R}^n)$.

Now we show that $\bar{\mathscr{A}}_0$ is, in fact, compact in $\bar{B}$. For each positive α, define $\bar{y}^\alpha \in C([-\alpha, 0], \mathbb{R}^n)$ as $\bar{y}^\alpha(\theta) = \bar{y}(\theta)$ for $\theta \in [-\alpha, 0]$. Since $\bar{\mathscr{A}}_0$ is equicontinuous and equibounded in $C((-\infty, 0], \mathbb{R}^n)$, we know that the set $\bar{\mathscr{A}}_0^\alpha = \{\bar{y}^\alpha: \bar{y} \in \bar{\mathscr{A}}_0\}$ is compact in $C([-\alpha, 0], \mathbb{R}^n)$. Therefore, for all $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ and functions $\bar{y}_1^\alpha, ..., \bar{y}_n^\alpha$, all contained in $\bar{\mathscr{A}}_0^\alpha$, such that for each $\bar{y}^\alpha \in \bar{\mathscr{A}}_0^\alpha$, there is a $\bar{y}_j^\alpha$ satisfying

$$\sup_{-\alpha \leqslant \theta \leqslant 0} |\bar{y}^\alpha(\theta) - \bar{y}_j^\alpha(\theta)| \leqslant \frac{\varepsilon}{2}.$$

Clearly, for each j, there is a $\bar{y}_j \in \bar{\mathscr{A}}_0$ such that $\bar{y}_j|_{[-\alpha,0]} = \bar{y}_j^{\alpha}$. Then, for any $y \in \bar{\mathscr{A}}_0$, we have, using the inequality in (A1) and the fact that $[\bar{T}(-\alpha)\,\bar{y}](\theta) = \bar{y}(-\alpha+\theta)$,

$$\begin{aligned}\|\bar{y}-\bar{y}_j\|_{\bar{B}} &= \|\bar{T}(\alpha)\,\bar{T}(-\alpha)\,\bar{y} - \bar{T}(\alpha)\,\bar{T}(-\alpha)\,\bar{y}_j\|_{\bar{B}}\\ &\leqslant K \sup_{0\leqslant s\leqslant\alpha} |\bar{T}(-\alpha)\,\bar{y}(s) - \bar{T}(-\alpha)\,\bar{y}_j(s)|\\ &\quad + M(\alpha)\,\|\bar{T}(-\alpha)\,\bar{y} - \bar{T}(-\alpha)\,\bar{y}_j\|_{\bar{B}}\\ &\leqslant K \sup_{-\alpha\leqslant\theta\leqslant 0} |y(\theta) - \bar{y}_j(\theta)| + M(\alpha)\,K_0\\ &= K \sup_{-\alpha\leqslant\theta\leqslant 0} |\bar{y}^{\alpha}(\theta) - \bar{y}_j^{\alpha}(\theta)| + M(\alpha)\,K_0.\end{aligned}$$

Therefore, we can find n, j, and an α large enough to make

$$\|\bar{y}-\bar{y}_j\|_{\bar{B}} < \varepsilon.$$

Hence we can cover $\bar{\mathscr{A}}_0$ by n balls with radius ε and center $\bar{y}_j$ and so $\bar{\mathscr{A}}_0$ is compact in $\bar{B}$.

LEMMA 5.5. *$\bar{\mathscr{A}}_0$ is invariant*

Proof. The proof is the same as in the case of parameter dependent finite delay problems.

Finally, we must show that

LEMMA 5.6. *$\bar{\mathscr{A}}_0$ attracts bounded sets of $\bar{B}$ in the $\bar{B}$-topology.*

Proof. Let $\bar{\Gamma}$ be a bounded subset of $\bar{B}$. Fix ε. From the inequality (2.1), we know that orbits of bounded sets under the flow $(1.1)_0$ are bounded in $\bar{B}$. Therefore, there is a constant $K_{\gamma(\bar{\Gamma})\cup\bar{\mathscr{A}}_0}$ such that for all $x \in \gamma(\bar{\Gamma}) \cup \bar{\mathscr{A}}_0$, $\|x\|_{\bar{B}} \leqslant K_{\gamma(\bar{\Gamma})\cup\bar{\mathscr{A}}_0}$. We can choose a constant $a>0$ such that $M(a) < (\varepsilon/2K_{\gamma(\bar{\Gamma})\cup\bar{\mathscr{A}}_0})$. Of course, $\Gamma = \{\bar{\phi}|_{[-r(0),0]} : \bar{\phi} \in \bar{\Gamma}\}$ is a bounded subset in B_0, so, from Theorem 3.3 (continuity with respect to initial conditions), there exists a $\delta(a,\varepsilon)$ such that for all $u, v \in \gamma(\Gamma) \cup \mathscr{A}_0$, $\|T(t)u - T(t)v\|_{B_0} < (\varepsilon/2K)$ for all $t \in [0,a]$ if $\|u-v\|_{B_0} < \delta$. Using the invariance of $\bar{\mathscr{A}}_0$ and the axioms (A1) through (A5), we can estimate the distance from $\bar{\mathscr{A}}_0$ to $\bar{T}(t)\bar{\Gamma}$. Let $\bar{\phi} \in \bar{\Gamma}$. Then

$$\begin{aligned}&\inf_{y\in\bar{\mathscr{A}}_0} \|\bar{T}(t)\bar{\phi} - \bar{y}\|_{\bar{B}}\\ &\quad = \inf_{y\in\bar{\mathscr{A}}_0} \|\bar{T}(a)\,\bar{T}(t-a)\bar{\phi} - \bar{T}(a)\,\bar{T}(t-a)\,\bar{y}\|_{\bar{B}}\\ &\quad \leqslant \inf_{y\in\bar{\mathscr{A}}_0} K \sup_{0\leqslant s\leqslant a} |\bar{T}(s)\,\bar{T}(t-a)\,\bar{\phi}(0) - \bar{T}(s)\,\bar{T}(t-a)\,\bar{y}(0)|\end{aligned}$$

$$
\begin{aligned}
&+ M(a)\,\|\bar{T}(t-a)\bar{\phi} - \bar{T}(t-a)\,\bar{y}\|_{\bar{B}} \\
&\leqslant \inf_{y\in\mathscr{A}_0} K \sup_{0\leqslant s\leqslant a} |T(s)\,T(t-a)\,\phi(0) - T(s)\,T(t-a)\,y(0)| \\
&\quad + M(a)\,K_{\gamma(\bar{\Gamma})\cup\bar{\mathscr{A}}_0} \\
&\leqslant \inf_{y\in\mathscr{A}_0} K \sup_{0\leqslant s\leqslant a} \|T(s)\,T(t-a)\phi - T(s)\,T(t-a)\,y\|_{B_0} \\
&\quad + M(a)\,K_{\gamma(\bar{\Gamma})\cup\bar{\mathscr{A}}_0}.
\end{aligned}
$$

Since $\mathscr{A}_0$ attracts bounded sets of B_0 in the B_0-topology, we can find a $\tau(\bar{\Gamma})>0$, independent of $\bar{\phi}$, such that for each $t>\tau+a$ there exists a $y^*\in\mathscr{A}_0$ so that $\|T(t-a)\phi - T(t-a)\,y^*\|_{B_0}<\delta$. Fix $t>\tau+a$ and hence y^*. Then, if we let u above be $T(t-a)\phi$ and v be $T(t-a)\,y^*$, we have

$$\|T(s)\,T(t-a)\phi - T(s)\,T(t-a)\,y^*\|_{B_0}\leqslant\varepsilon$$

for all $s\in[0,a]$, and so

$$
\begin{aligned}
&\inf_{\bar{y}\in\bar{\mathscr{A}}_0} \|\bar{T}(t)\bar{\phi}-\bar{y}\|_{\bar{B}} \\
&\qquad\leqslant K \sup_{0\leqslant s\leqslant a} \|T(s)\,T(t-a)\phi - T(s)\,T(t-a)\,y^*\|_{B_0} + M(a)K\leqslant\varepsilon
\end{aligned}
$$

We get this estimate for each $t>\tau+a$. Therefore, since this holds for all $\bar{\phi}\in\bar{\Gamma}$, we have

$$\sup_{\bar{\phi}\in\bar{\Gamma}}\ \inf_{\bar{y}\in\bar{\mathscr{A}}_0} \|\bar{T}(t)\bar{\phi}-\bar{y}\|_{\bar{B}}\leqslant\varepsilon$$

for all $t>\tau+a$ and so $\bar{\mathscr{A}}_0$ attracts bounded sets under the flow $(1.1)_0$ in $\bar{B}$.

Thus we have proven the Theorem 4.1.

6. Proof of Theorems 4.2 and 4.3

Here we only give the proof in the context of Theorem 4.3 with occasional remarks about the application to Theorem 4.2. If $f_\lambda\to f_0$ uniformly in bounded sets, then $f_\lambda\to f_0$ uniformly in C_r-equicontinuous sets for $r=0$ and the conditions (c1) through (c3) are satisfied for $r=0$. Therefore, Theorem 4.2 is just a special case of Theorem 4.3.

Let $\bar{\Gamma}\subset\bar{B}$ be a bounded set. First we will show that we have convergence of the solutions, $\bar{x}_t(\cdot\,;\phi,\lambda)$, of $(1.1)_\lambda$ to solutions, $\bar{x}_t(\cdot\,;\phi,\lambda)$, of $(1.1)_0$ uniformly for t in compact intervals and for ϕ in the C_r-equicontinuous set $F(r,\bar{\Gamma})$; that is, for any t_0, we can find a $\hat{\lambda}$ which depends on t_0 and on F such that, for all $\phi\in F$ and for all $t\in[0,t_0]$, $\|\bar{x}_t(\cdot\,;\phi,\lambda)-\bar{x}_t(\cdot\,;\phi,0)\|_{\bar{B}}\to 0$

as $\lambda \to 0$. Then we can use the fact that the flow is autonomous, that is, the semigroup property, to step forward by intervals of length t_0. Since the set $\bigcup_\lambda \bar{T}_\lambda(r)\bar{\Gamma}$ is C_r-equicontinuous, this, along with condition (c1), proves that orbits of bounded sets are bounded. Therefore, since each $\bar{T}_\lambda(t)$ is an α-contraction in $\bar{B}$, we know that ω-limit sets exist. The argument also implies then that, for small λ, $\omega_\lambda(\bigcup_\lambda \bar{T}_\lambda(r)\bar{\Gamma})$, and therefore $\omega_\lambda(\bar{\Gamma})$, is contained in an arbitrarily small neighborhood of the attractor $\mathscr{A}_0$. Hence we have upper semicontinuity.

We begin with the Lemma

LEMMA 6.1. *Let $\bar{\Gamma} \subset \bar{B}$ be bounded. Suppose conditions* (c1) *through* (c3) *are satisfied and let $\phi \in F(r, \bar{\Gamma})$. Then for each ε and t_0 there is a $\hat{\lambda}(\varepsilon, t_0, F)$ such that for all $\lambda \leqslant \hat{\lambda}$ there is a unique solution, $\bar{x}_t(\cdot; \bar{\phi}, \lambda)$, which exists for $t \in [0, t_0]$ and which satisfies $\|\bar{x}_t(\cdot; \bar{\phi}, \lambda) - \bar{x}_t(\cdot; \bar{\phi}, 0)\|_{\bar{B}} \leqslant \varepsilon$ uniformly for $t \in [0, t_0]$ and uniformly in F.*

Proof. Define $W_0 = \{\bar{x}_t(\cdot, \bar{\phi}, 0) : \bar{\phi} \in F, t \in [0, t_0]\}$. Then W_0 is bounded and, in fact, it is C_r-equicontinuous. Since the functions f_λ are bounded, uniformly in λ, on bounded sets, we can define $A(\alpha, \xi)$, $V_0 = V(\alpha, \xi)$ and $M(\xi)$ as in Lemma 3.1 with $M(\xi)$, α and ξ chosen so that $M(\xi)\alpha < \xi/2$ and $\alpha l(v)K < 1$. From Theorem 3.2 there exists a $\tilde{\lambda} > 0$ such that for each $\lambda \leqslant \tilde{\lambda}$ and for each $\bar{\phi} \in F$, there is a unique solution, $\bar{x}_t(\cdot, \bar{\phi}, \lambda)$, which exists for $t \in [0, \alpha]$.

From Theorem 3.4, if $f_\lambda \to f_0$ uniformly for $\bar{\phi} \in F$, then $\bar{x}_t(\cdot; \bar{\phi}, \lambda) \to \bar{x}_t(\cdot; \bar{\phi}, 0)$ uniformly for $\bar{\phi} \in F$ and for $t \in [0, \alpha]$. In fact, for $t \in [0, \alpha]$,

$$
\begin{aligned}
&\|\bar{x}_t(\cdot; \bar{\phi}, \lambda) - \bar{x}_t(\cdot; \bar{\phi}, 0)\|_{\bar{B}} \\
&\quad \leqslant \frac{K}{1 - \alpha l(V)K} \int_0^\alpha |f_\lambda(\hat{\phi}_s + g(\bar{\phi}, f_0)_s) - f_0(\hat{\phi}_s + g(\bar{\phi}, f_0)_s)|\, ds \\
&\quad \leqslant \frac{K\alpha}{1 - \alpha l(V)K} \sup_{\phi \in \tilde{V}_0} |f_\lambda(\phi) - f_0(\phi)| \qquad (6.1)
\end{aligned}
$$

where $\tilde{V}_0 = \{\hat{\phi}_s + y_s : \bar{\phi} \in W_0, y \in \operatorname{Im} G; s \in [0, \alpha]\} \subset V$. G is the map defined in section 3. $\tilde{V}$ is C_r-equicontinuous since W_0 is and since the image of G is compact (see section 3). Therefore, we can choose $\tilde{\lambda}^\alpha \leqslant \tilde{\lambda}$ so that $(K\alpha/1 - \alpha l(V)K) \sup_{\phi \in \tilde{V}_0} |f_\lambda(\phi) - f_0(\phi)| < \varepsilon/2$; then we'll have $\|\bar{x}_\alpha(\cdot; \bar{\phi}, \lambda) - \bar{x}_\alpha(\cdot; \bar{\phi}, 0)\|_{\bar{B}} < \varepsilon/2$ for all $\lambda \leqslant \tilde{\lambda}^\alpha$. Then $\bar{x}_\alpha(\cdot; \bar{\phi}, \lambda) \in V_0$ for all $\lambda \leqslant \tilde{\lambda}^\alpha$ since $\bar{x}_\alpha(\cdot; \bar{\phi}, \lambda) = \bar{x}_\alpha(\cdot; \bar{\phi}, 0) + \bar{x}_\alpha(\cdot; \bar{\phi}, \lambda) - \bar{x}_\alpha(\cdot; \bar{\phi}, 0)$. Define $W_\alpha = W_0 \cup \{\bar{x}_\alpha(\cdot; \bar{\phi}, \lambda) : \lambda \leqslant \tilde{\lambda}^\alpha, \bar{\phi} \in F\}$ and define $V_\alpha = \{\phi + \psi : \phi \in W_\alpha \cup U(\alpha, \xi/2), \psi \in \bar{B}$ and $\|\psi\|_{\bar{B}} < \xi/2\}$. Then $V_\alpha \subset V_0$ since if $\phi \in W_0$, then, clearly, $\phi + \psi \in V_0$, and if $\phi \in \{\bar{x}_\alpha(\cdot; \bar{\phi}, \lambda) : \lambda \leqslant \tilde{\lambda}^\alpha, \bar{\phi} \in F\}$ then $\phi + \psi = \bar{x}_\alpha(\cdot; \bar{\phi}, 0) + \bar{x}_\alpha(\cdot; \bar{\phi}, \lambda) - \bar{x}(\cdot; \bar{\phi}, 0) + \psi$ and $\|\bar{x}_\alpha(\cdot; \bar{\phi}, \lambda) - \bar{x}(\cdot; \bar{\phi}, 0) + \psi\| < \xi$. Therefore,

$|f_\lambda(\phi)| < M(\xi)$ for $\lambda \leqslant \tilde{\lambda}^\alpha$ and $\phi \in V_\alpha$ and the map G: $W_\alpha \times \{f_\lambda; \lambda \leqslant \tilde{\lambda}^\alpha\} \times A(\alpha, \xi) \to A(\alpha, \xi)$ has a fixed point in $A(\alpha, \xi)$ for each $\phi \in W_\alpha$ and $\lambda \leqslant \tilde{\lambda}^\alpha$. That is, for $\lambda \leqslant \tilde{\lambda}^\alpha$ and $\phi \in W_\alpha$ a unique solution of $(1.1)_\lambda$ through ϕ exists on the interval $[0, \alpha]$. Also for $t \in [0, \alpha]$ and $\phi \in W_\alpha$,

$$\|\bar{x}_t(\cdot\,; \phi, \lambda) - \bar{x}_t(\cdot\,; \phi, 0)\|_{\bar{B}} \leqslant \frac{K\alpha}{1 - \alpha l(V)K} \sup_{\phi \in \tilde{V}_\alpha} |f_\lambda(\phi) - f_0(\phi)|$$

where $\tilde{V}_\alpha = \{\hat{\phi}_s + y_s \colon \phi \in W_\alpha, y \in \operatorname{Im} G; s \in [0, \alpha]\} \subset \tilde{V}_0$ is C_r-equicontinuous. Then, for $\bar{\phi} \in F$ and $t \in [\alpha, 2\alpha]$, we have, using the estimates in Theorem 3.3

$$\begin{aligned}
&\|\bar{x}_t(\cdot\,; \bar{\phi}, \lambda) - \bar{x}_t(\cdot\,; \bar{\phi}, 0)\|_{\bar{B}} \\
&\quad = \|\bar{x}_{t-\alpha}(\cdot\,; \bar{x}_\alpha(\cdot\,; \bar{\phi}, \lambda), \lambda) - \bar{x}_{t-\alpha}(\cdot\,; \bar{x}_\alpha(\cdot\,; \bar{\phi}, 0), 0)\|_{\bar{B}} \\
&\quad \leqslant \|\bar{x}_{t-\alpha}(\cdot\,; \bar{x}_\alpha(\cdot\,; \bar{\phi}, \lambda), \lambda) - \bar{x}_{t-\alpha}(\cdot\,; \bar{x}_\alpha(\cdot\,; \bar{\phi}, \lambda), 0)\|_{\bar{B}} \\
&\qquad + \|\bar{x}_{t-\alpha}(\cdot\,; \bar{x}_\alpha(\cdot\,; \bar{\phi}, \lambda), 0) - \bar{x}_{t-\alpha}(\cdot\,; \bar{x}_\alpha(\cdot\,; \bar{\phi}, 0), 0)\|_{\bar{B}} \\
&\quad \leqslant \frac{K\alpha}{1 - \alpha l(V)K} \sup_{\phi \in \tilde{V}_\alpha} |f_\lambda(\phi) - f_0(\phi)| \\
&\qquad + \frac{\alpha l(V)(K + \bar{M})}{1 - \alpha l(V)K} \|\bar{x}_\alpha(\cdot\,; \bar{\phi}, \lambda) - \bar{x}_\alpha(\cdot\,; \bar{\phi}, 0)\|_{\bar{B}}
\end{aligned} \tag{6.2}$$

So, if we choose $\tilde{\lambda}^{2\alpha}$ so that $(K\alpha/(1 - \alpha l(V)K)) \sup_{\phi \in \tilde{V}} |f_\lambda(\phi) - f_0(\phi)| < \varepsilon/4$, then $\|\bar{x}_t(\cdot, \bar{\phi}, \lambda) - \bar{x}_t(\cdot, \bar{\phi}, 0)\|_{\bar{B}} < \varepsilon/2$ for all $\bar{\phi} \in F$, for all $t \in [0, 2\alpha]$ and for all $\lambda \leqslant \tilde{\lambda}^{2\alpha}$.

We continue in this manner until we have covered the entire interval $[0, t_0]$. Then if we choose $\hat{\lambda} = \min\{\tilde{\lambda}^{n\alpha} \colon n \leqslant t/\alpha + 1\}$, we have the result.

It is important to notice that, in general, we cannot choose $\hat{\lambda}$ above independently of the set F. If $f_\lambda \to f_0$ uniformly in $\bar{B}$, and so $F = \bar{\Gamma}$, the estimates in (6.1) and (6.2) are independent of $\bar{\Gamma}$. In that case, we can choose $\hat{\lambda}$ independently of the bounded set.

Next we use Theorem 4.1 and Lemma 6.1 to prove that if $\bar{\Gamma}$ is any arbitrarily large neighborhood of the attractor $\mathscr{A}_0$, then the flow of $F(r, \bar{\Gamma})$ under $(1.1)_\lambda$ enters an arbitrarily small neighborhood of the attractor. We will use the notation $\bar{N}(\delta, \bar{\Gamma})$ for the set $\{\bar{\phi} \colon d(\phi, \bar{\Gamma}) < \delta\}$ where $\bar{\Gamma} \subset \bar{B}$ and d is the distance in the $\bar{B}$-topology

Lemma 6.2. *Suppose hypotheses* (c1) *through* (c3) *hold. Let* $d > 0$ *be fixed. Define* $\bar{F} := F(r, \bar{N}(d, \mathscr{A}_0))$. *Then for all* δ_1, δ_2 *with* $\delta_1 + \delta_2 < d$, *there exists a* $\tilde{\lambda}$ *and* $t_1(\delta_2, d)$ *such that* $T_\lambda(t)\bar{F} \subset N(\delta_1 + \delta_2, \mathscr{A}_0)$ *for all* $\lambda \leqslant \tilde{\lambda}$ *and* $t > t_1(\delta_2, d)$.

Proof. Since $\mathscr{A}_0$ is an attractor for the flow $(1.1)_0$, we know that for all δ_2 there exists $\tau(\delta_2, d)$ such that $\bar{T}_0(t)\bar{F} \subset \bar{N}(\delta_2, \mathscr{A}_0)$ for all $t > \tau(\delta_2, d)$. Without loss of generality, we assume $\tau > r$. We can apply Lemma 6.1 with $t_0 = 2\tau(\delta_2, d) + r$ and $\varepsilon = \delta_1$. Keeping in mind that the solutions of $(1.1)_\lambda$ are unique, we get $\bar{T}_\lambda(t)\bar{F} \subset \bar{N}(\delta_1, T_0(t)\bar{F})$ for $\lambda \leqslant \hat{\lambda}(\delta_1, 2\tau + r, d)$ and $0 < t < 2\tau + r$. We remark that if $r = 0$ and condition (U) is satisfied, then $\hat{\lambda}$ will no longer depend on d. By the triangle inequality,

$$\bar{T}_\lambda(t)\bar{F} \subset \bar{N}(\delta_1 + \delta_2, \mathscr{A}_0) \qquad \text{for} \quad \tau < t < 2\tau + r \quad \text{and} \quad \lambda \leqslant \hat{\lambda}$$

If we apply the semigroup $\bar{T}_\lambda(s)$ to both sides then $\bar{T}_\lambda(s)\,\bar{T}_\lambda(t)\bar{F} \subset \bar{T}_\lambda(s)\,\bar{N}(\delta_1 + \delta_2, \mathscr{A}_0)$, but

$$\begin{aligned}
\bar{T}_\lambda(s)\,\bar{N}(\delta_1 + \delta_2, \mathscr{A}_0) &= \bar{T}_\lambda(s - r)\,\bar{T}_\lambda(r)\,\bar{N}(\delta_1 + \delta_2, \mathscr{A}_0) \\
&\subset \bar{T}_\lambda(s - r)\,F(r, \bar{N}(\delta_1 + \delta_2, \mathscr{A}_0)) \\
&\subset \bar{T}_\lambda(s - r)\bar{F} \\
&\subset N(\delta_1 + \delta_2, \mathscr{A}_0) \quad \text{for } s - r \in [\tau, 2\tau + r] \text{ and } \lambda \leqslant \hat{\lambda}
\end{aligned}$$

so

$$\bar{T}_\lambda(t)\bar{F} \subset N(\delta_1 + \delta_2, \mathscr{A}_0) \qquad \text{for} \quad \lambda \leqslant \hat{\lambda}$$

$$\tau < t + s \leqslant 4\tau + 3r$$

We continue in this manner to prove the lemma.

With this, we can prove the Theorems 4.2 and 4.3.

Proof. Fix $\varepsilon > 0$. Suppose $\bar{\Gamma} \subset \bar{B}$ is bounded and let d be such that $\bar{\Gamma} \subset U := N(d, \mathscr{A}_0)$. Define $\bar{F} = F(r, U)$. Since $\bar{F}$ is C_r-equicontinuous, Lemmas 6.1 and 6.2 imply that the orbit $\{T_\lambda(t)\bar{F}: t > 0\}$ is bounded. Since $T_\lambda(t)$ is a conditional α-contraction, this implies that $\omega_\lambda(\bar{F})$ exists and is compact and invariant. From Lemma 6.2, there is a $\hat{\lambda}(\bar{F})$ such that for all $\lambda \leqslant \hat{\lambda}$, $\omega_\lambda(\bar{\Gamma}) \subset \omega_\lambda(U) = \omega_\lambda(\bar{F}) \subset N(\varepsilon, \mathscr{A}_0)$. If we define $\mathscr{A}_\lambda := \omega_\lambda(\bar{F})$, then $\mathscr{A}_\lambda$ is a local attractor and the family $\{\mathscr{A}_\lambda\}$ is upper semicontinuous at $\lambda = 0$.

If the hypothesis (UBD) is satisfied, there exists an $R > 0$, independent of U, such that $\bar{T}_\lambda(t)(U)$ eventually enters $\bar{b}(R)$. Define $\bar{F} := F(r, \bar{b}(R))$. Since $\bar{F}$ is C_r-equicontinuous, again Lemmas 6.1 and 6.2 imply that the orbit $\{T_\lambda(t)\bar{F}: t > 0\}$ is bounded. Since $T_\lambda(t)$ is a conditional α-contraction, this implies that $\omega_\lambda(\bar{F})$ exists and is compact and invariant. From Lemma 6.2, there exists a $\hat{\lambda}(R)$ such that for all $\lambda \leqslant \hat{\lambda}$, $\omega_\lambda(\bar{b}(R)) \subset \bar{N}(\varepsilon, \mathscr{A}_0)$, but for every bounded set $\Gamma \subset \bar{B}$, $\omega_\lambda(\bar{\Gamma}) \subset \omega_\lambda(\bar{b}(R)) \subset \omega_\lambda(\bar{F}) \subset \bar{N}(\varepsilon, \mathscr{A}_0)$. If we define $\mathscr{A}_\lambda := \omega_\lambda(\bar{F})$), then $\mathscr{A}_\lambda$ is a global attractor and the family $\{\mathscr{A}_\lambda\}$ is upper semicontinuous at $\lambda = 0$.

If $f_\lambda \to f_0$ uniformly in bounded sets then $r = 0$ and $\bar{F} = \bar{\Gamma}$ and we have existence of local attractors $\{\mathscr{A}_\lambda\}$ and upper semicontinuity at $\lambda = 0$. The attractors are global if condition (UBD) holds. If condition (U) holds, then $\hat{\lambda}$ is independent of $\bar{\Gamma}$ and, again, the attractors are global.

7. Approximation of Continuous Delays by Discrete Delays

In this section, we will apply the results of the previous sections to the example (1.7) and $(1.8)_n$ in the introduction. The techniques in the example can be applied to other problems where the limiting equation has discrete delays. Problems where all the memory functions are continuous will be addressed in a forthcoming paper (Hines [1992]).

We will make rigorous the idea of May [1973] that the models

$$\dot{N}(t) = rN(t)\left(1 - \int_{-\infty}^{t} N(\theta)\, Q_n(t-\theta)\, d\theta\right) := f_n(N_t) \qquad (7.1)_n$$

should in some sense converge to the model

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t-T)}{K}\right) := f(N_t) \qquad (7.1)$$

where $Q_1(t) = Q(t) = (1/KT)(t/T)\, e^{(-t/T)}$ and the sequence $Q_n(t)$ converges in some sense to the δ-function. We will construct a sequence $\{Q_n\}$, given $Q(t)$, for which we have that $f_n \to f$ as $n \to \infty$ uniformly on C_r-equicontinuous sets, where r is any real number strictly larger than 1. Since the problem (7.1) has a global attractor in the subspace $\{\phi \in C([-1, 0], \mathbb{R})$: $\phi(\theta) > 0, \theta \in [-1, 0]\} \subset C([-1, 0], \mathbb{R})$, we will be able to apply the results of section 6 to prove that, for large n, there is a family of local attractors $\{\mathscr{A}_n\}$ for the problems $(7.1)_n$ which is upper semicontinuous at $n = \infty$ and which satisfies the property (A). We will also be able to verify the condition (UBD) and so the attractors will be global.

We begin by showing how to construct a sequence $\{k_n\}$, given any k: $(-\infty, 0] \to \mathbb{R}$ which is piecewise continuous and any constant $T > 0$, in such a way that $k_1 = k$ and $\{k_n\}$ satisfies the weak convergence property

$(\mathrm{WC})_T$ If ψ: $\mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then

$$\lim_{n \to \infty} \int_{-\infty}^{0} k_n(\theta)\, \psi(\theta)\, d\theta = \int_{-\infty}^{0} dH(\theta)\, \psi(\theta) = \psi(-T)$$

where $H(\theta)$ is the Heaviside function which takes on the value 1 for $\theta \geqslant -T$ and the value 0 for $\theta < -T$.

For this we will need the concept of a Dirac sequence (see Figueiredo [1977]). We will say that a sequence $\{k_n\}$ of functions defined on $(-\infty, 0]$ is a *Dirac sequence* if it is a sequence of piecewise continuous functions which satisfies the following properties for some $T > 0$.

(d1) For each n and for each $\theta \in (-\infty, 0]$, $k_n(\theta) > 0$

(d2) For each n, $\int_{-\infty}^{0} k_n(\theta)\, d\theta = 1$

$(\mathrm{d3})_T$ For each $\varepsilon > 0$ and $\delta > 0$, there is an n_0 such that $\int_{|\theta-(-T)|>\delta} k_n(\theta)\, d\theta < \varepsilon$ for all $n \geqslant n_0$

We then have the following Lemma

LEMMA 7.1. *A Dirac sequence which satisfies properties* (d1) *through* $(\mathrm{d3})_T$ *also satisfies the weak convergence property* $(\mathrm{WC})_T$.

Proof. To show the convergence, we consider the difference

$$\begin{aligned}
&\int_{-\infty}^{0} k_n(\theta)\, \psi(\theta)\, d\theta - \psi(-T) \\
&\quad = \int_{-\infty}^{0} k_n(\theta)(\psi(\theta) - \psi(-T))\, d\theta \\
&\quad = \int_{-\infty}^{-T-\delta} k_n(\theta)(\psi(\theta) - \psi(-T))\, d\theta + \int_{-T-\delta}^{-T+(T/n)} k_n(\theta)(\psi(\theta) - \psi(-T))\, d\theta \\
&\quad := I_1 + I_2
\end{aligned}$$

For any given ε, we can find an n_1 and a δ so that $|\psi(\theta) - \psi(-T)| < \varepsilon$ for all $\theta \in [-T-\delta, -T+(T/n)]$. In that case, we have $|I_2| < \varepsilon$ for all $n \geqslant n_1$. Since $(\mathrm{d3})_T$ is satisfied, we know that for any ε and δ, there is an n_2 such that $\int_{-\infty}^{-T-\delta} k_n(\theta)\, d\theta < (\varepsilon/M)$ for all $n \geqslant n_2$, where $M > |\psi(\theta)|$ for all θ. Then we have $|I_1| < \varepsilon$. Therefore, for each ε, there exists an $\bar{n} = \max\{n_1, n_2\}$, such that

$$\left| \int_{-\infty}^{0} k_n(\theta)\, \psi(\theta)\, d\theta - \psi(-T) \right| < 2\varepsilon$$

for all $n \geqslant \bar{n}$ and hence the convergence property $(\mathrm{WC})_T$ is satisfied.

Given any piecewise continuous function $k\colon (-\infty, 0] \to \mathbb{R}$ which satisfies (d1) and (d2) and any positive constant $T > 0$, it is easy to construct a Dirac sequence with $k_1 = k$. We define

$$k_n(\theta) = \begin{cases} nk_1(n(\theta+T) - T) & \theta \leqslant -T + (T/n) \\ 0 & \theta > -T + (T/n) \end{cases}$$

It is straightforward to show that the sequence $\{k_n\}$ satisfies conditions (d1) through $(\mathrm{d}3)_T$.

Now we can apply this to our example. To facilitate the analysis, we first rescale the time in (7.1) so that the problem has a unit delay; that is, we define $y(t) = N(tT)$. Then $y(t)$ satisfies the differential equation

$$\dot{y}(t) = rT\, y(t) - \frac{rT}{K}\, y(t-1)\, y(t).$$

If we then make the change of variables, $x(t) = (1/K)\, y(t) - 1$, this equation becomes

$$\dot{x}(t) = -rT\, x(t-1)(1 + x(t)). \tag{7.2}$$

We can apply the same transformations to $(7.1)_n$. If we do, we obtain the differential equation

$$\dot{x}(t) = -rT \int_{-\infty}^{t} x(\theta)\, \bar{Q}(t-\theta)\, d\theta)(1 + x(t)) \tag{7.3}$$

where $\bar{Q}_n(t) = KTQ_n(tT)$ and $\bar{Q}_1(t) = \bar{Q}(t) = te^{-t}$. To make the notation consistent with the previous sections, we define $\bar{\bar{Q}}\colon \mathbb{R}^- \to \mathbb{R}$ by $\bar{\bar{Q}}(t) = \bar{Q}(-t)$. In the sequel, we will leave off the bars and just write $Q(t)$. So finally we have

$$\dot{x}(t) = -rT \int_{-\infty}^{t} x(\theta)\, Q(t-\theta)\, d\theta)(1 + x(t)) \tag{7.4}$$

where $Q_1(t) = Q(t) = -te^t$ and $Q(t)$ has a bump at the discrete delay -1.

The sequence $\{Q_n\}$ will be a Dirac sequence which we construct so that $Q_1 = Q$ and $T = 1$, according to the previous discussion; that is

$$Q_n(\theta) = \begin{cases} nQ(n(\theta+1)-1) & \theta \leqslant -1 + (1/n) \\ 0 & \theta > -1 + (1/n). \end{cases}$$

Besides satisfying conditions (d1) through $(\mathrm{d}3)_1$ and the convergence property $(\mathrm{WC})_1$, this sequence also satisfies

PROPOSITION 7.2. *The quotient* $(Q_n(\theta)/Q_1(\theta))$ *satisfies*

$$\sup_{-\infty \leqslant \theta \leqslant -T-1} (Q_n(\theta)/Q_1(\theta)) < \infty$$

uniformly in n, *and, for each fixed* $\delta > 0$,

$$\sup_{-\infty \leqslant \theta \leqslant -T-\delta} (Q_n(\theta)/Q_1(\theta)) \to 0 \qquad \textit{as} \quad n \to \infty.$$

Proof

$$\frac{Q_n(\theta)}{Q(\theta)} = \frac{n(n(\theta+1)-1)\, e^{n(\theta+1)-1}}{\theta e^{\theta}}$$

$$= \frac{(n^2\theta + n^2 - n)}{\theta}\, e^{(n-1)(\theta+1)}$$

$$\leqslant \left(n^2 + \frac{n^2}{|\theta|} + \frac{n}{|\theta|}\right) e^{(n-1)(\theta+1)}$$

so

$$\sup_{-\infty \leqslant \theta \leqslant -2} \frac{Q_n(\theta)}{Q(\theta)} \leqslant \left(n^2 + \frac{n^2}{2} + \frac{n}{2}\right) e^{-(n-1)} \leqslant (2n^2)\, e^{-(n-1)} \leqslant 8e^{-1}$$

and

$$\sup_{-\infty \leqslant \theta \leqslant -1-\delta} \frac{Q_n(\theta)}{Q(\theta)} \leqslant \left(n^2 + \frac{n^2}{1+\delta} + \frac{n}{1+\delta}\right) e^{-\delta(n-1)}$$

$$\leqslant (2n^2 + n)\, e^{-\delta(n-1)}$$

which goes to 0 as n goes to infinity.

Now we can study the problem

$$\dot{x}(t) = -rT \int_{-\infty}^{t} x(\theta)\, Q_n(t-\theta)\, d\theta \left(1 + x(t)\right) := f_n(x_t) \qquad (7.6)_n$$

along with the limiting problem

$$\dot{x}(t) = -rT\, x(t-1)(1 + x(t)) := f_{\infty}(x_t). \qquad (7.6)_{\infty}$$

The natural phase space in which to consider $(7.6)_{\infty}$ is of course $C([-1,0],\mathbb{R})$. After the transformation (7.2), this problem admits a global attractor in the subspace $C_{-1} := \{\phi \in C([-1,0],\mathbb{R}) : \phi(\theta) > -1$ for all $\theta \in [-1,0]\}$. From the discussion in section 2, we might consider $(7.6)_n$ in the space $B_n = \{\phi\colon (-\infty, 0] \to \mathbb{R}\colon \phi$ is measurable and $\|\phi\|_{B_n} < \infty\}$ where $\|\phi\|_{B_n} = \int_{-\infty}^{0} |\phi(\theta)|\, Q_n(\theta)\, d\theta$. These spaces are nested; that is $B_n \subset B_m$ if $n < m$. In particular,

PROPOSITION 7.3. *$B_1 \subset B_n$ for all n.*

Proof. Suppose $\phi \in B_1$, then

$$\begin{aligned}
\|\phi\|_{B_n} &= \int_{-\infty}^{0} |\phi(\theta)|\, Q_n(\theta)\, d\theta \\
&\leqslant \sup_{-2 \leqslant \theta \leqslant 0} |\phi(\theta)| + \int_{-\infty}^{-2} |\phi(\theta)|\, Q_n(\theta)\, d\theta \\
&\leqslant \sup_{-2 \leqslant \theta \leqslant 0} |\phi(\theta)| + \int_{-\infty}^{-2} |\phi(\theta)|\, Q_1(\theta)\, d\theta \cdot \sup_{-\infty \leqslant \theta \leqslant -2} \left\{\frac{Q_n(\theta)}{Q_1(\theta)}\right\} \\
&\leqslant \sup_{-2 \leqslant \theta \leqslant 0} |\phi(\theta)| + 8e^{-1} \int_{-\infty}^{-2} |\phi(\theta)|\, Q_1(\theta)\, d\theta \\
&\leqslant (1 + 8e^{-1})\, \|\phi\|_{\bar{B}}
\end{aligned}$$

Therefore, according to the discussion in section 2, we will consider all the problems $(7.6)_n$, $n \geqslant 0$ in the space $\bar{B} := \{\phi : (-\infty, 0] \to \mathbb{R} : \phi$ is measurable and $\phi_{[-2,0]}$ is continuous and $\|\phi\|_{\bar{B}} < \infty\}$ where $\|\phi\|_{\bar{B}} = \sup_{-2 \leqslant \theta \leqslant 0} |\phi(\theta)| + \int_{-\infty}^{-2} |\phi(\theta)|\, Q_1(\theta)\, d\theta$.

We can now state the Theorem.

THEOREM 7.4. *For each n large enough, the problem* $(7.6)_n$ *admits a global attractor* $\mathscr{A}_n$ *for the flow considered in the subspace* $\bar{B}_{-1} := \{\phi \in \bar{B} : \phi(\theta) > -1$ *for all* $\theta \in (-\infty, 0]\}$ *of* $\bar{B}$. *Furthermore, the attractors are upper semicontinuous at* $n = \infty$.

To prove this Theorem, we will prove that the flow $(1.1)_n$ is invariant in $\bar{B}_{-1}$ and that conditions (L), (UBD), and (c1) through (c3) hold for the problem $(7.6)_n$. Then Theorem 7.4 is just a simple application of Theorem 4.3

It is easy to prove that the f_n satisfy condition (L); that is

PROPOSITION 7.5. *If* Γ *is a bounded set in* $\bar{B}$, *then* f_n *is Lipschitz continuous on* Γ *with Lipshitz constant* $l_n(\Gamma)$ *and* $\sup_n l_n(\Gamma)$ *is finite.*

Proof. Let $\phi, \psi \in \Gamma$. Then

$$\begin{aligned}
&|f_n(\phi) - f_n(\psi)| \\
&= \left| -rT \int_{-\infty}^{0} \phi(\theta)\, Q_n(\theta)\, d\theta \left(1 + \phi(0)\right) + rT \int_{-\infty}^{0} \psi(\theta)\, Q_n(\theta)\, d\theta \left(1 + \psi(0)\right) \right| \\
&= \left| rT \int_{-\infty}^{0} (\psi(\theta) - \phi(\theta))\, Q_n(\theta)\, d\theta \right.
\end{aligned}$$

$$+rT\left(\psi(0)\int_{-\infty}^{0}\psi(\theta)\,Q_n(\theta)\,d\theta-\phi(0)\int_{-\infty}^{0}\phi(\theta)\,Q_n(\theta)\,d\theta\right)\Bigg|$$

$$\leqslant rT\int_{-\infty}^{0}|\psi(\theta)-\phi(\theta)|\,Q_n(\theta)\,d\theta$$

$$+rT\left|\psi(0)\int_{-\infty}^{0}\psi(\theta)\,Q_n(\theta)\,d\theta-\psi(0)\int_{-\infty}^{0}\phi(\theta)\,Q_n(\theta)\,d\theta\right|$$

$$+rT\left|\psi(0)\int_{-\infty}^{0}\phi(\theta)\,Q_n(\theta)\,d\theta-\phi(0)\int_{-\infty}^{0}\phi(\theta)\,Q_n(\theta)\,d\theta\right|$$

$$\leqslant rT(1+\|\psi\|)_{\bar{B}}\,\|\psi-\phi\|_{B_n}+rT\,\|\phi\|_{B_n}\,\|\psi-\phi\|$$

$$\leqslant rT(1+\|\psi\|_{\bar{B}})(1+8e^{-1})\,\|\psi-\phi\|_{\bar{B}}+rT(1+8e^{-1})\,\|\phi\|_{\bar{B}}\,\|\psi-\phi\|_{\bar{B}}$$

$$\leqslant(1+8e^{-1})\,rT(1+C_\Gamma)\,\|\psi-\phi\|_{\bar{B}}$$

where $C_\Gamma>\|\phi\|_{\bar{B}}$ for all $\phi\in\Gamma$. So $l_n(\Gamma)=(1+8e^{-1})\,rT(1+C_\Gamma)$ and the proposition is proved.

Remark 7.6. Propositions 7.4 and 7.5 are results only of Proposition 7.2 and condition (L).

Next, we prove that the flow is invariant in $\bar{B}_{-1}$ and that hypothesis (UBD) and condition (c1) hold. In the proofs, we will need to use the inequality in axiom (A1). In $\bar{B}$, we can estimate the norm of the translation operator $S(t)$ by $\|S(t)\|\leqslant(1+t)\,e^{-t}:=\bar{M}(t)$, so the inequality can be given explicitly as

$$\|x_t\|_{\bar{B}}\leqslant 2\sup_{0\leqslant s\leqslant t}|x(s)|+(1+t)\,e^{-t}\,\|\phi\|_{\bar{B}}$$

though this is not the best estimate. For any real number $\Theta>0$, this can also be written as

$$\|x_t\|_{\bar{B}}=\|T(t-\Theta)\,T(\Theta)\phi\|_{\bar{B}}$$

$$\leqslant 2\sup_{0\leqslant s\leqslant t-\Theta}|T(\Theta)\,\phi(s)|+(1+t-\Theta)\,e^{-(t-\Theta)}\,\|T(\Theta)\phi\|_{\bar{B}}$$

$$\leqslant 2\sup_{\Theta\leqslant s\leqslant t}|x(s)|+(1+t-\Theta)\,e^{-(t-\Theta)}\,\|T(\Theta)\phi\|_{\bar{B}} \tag{7.7}$$

PROPOSITION 7.7. *For each problem* $(7.6)_n$, $n\geqslant 0$, *if* $\Gamma\subset\bar{B}_{-1}$ *is bounded and* $\phi\in\Gamma$, *then the orbit* $T_n(t)\phi$ *exists for all time and remains in* $\bar{B}_{-1}$ *and the flow* $T_n(t)\Gamma$ *eventually enters and remains in a ball,* $\bar{b}(R)$, *of radius* R *where* R *does not depend on* n.

Proof. First we consider the flow defined by $(7.6)_\infty$. If we integrate $(7.6)_\infty$, then we see that a solution $x(t)$ of $(7.6)_\infty$ also satisfies

$$x(t) = (1 + x(t_0)) \exp\left\{-rT \int_{t_0}^{t} x(s)\, ds\right\} - 1 \qquad (7.8)_\infty$$

so if $\phi(0) > -1$, then $x(t) > -1$ as long as it is defined. But equation $(7.8)_\infty$ also shows that $x(t)$ may not become unbounded in finite time. In fact,

$$x(t) < (1 + \phi(0)) \exp\left\{rT \int_{-1}^{t-1} ds\right\} - 1 < (1 + \phi(0)) \exp(rTt) - 1. \quad (7.9)$$

Therefore $T_0(t)\phi := x_t(\cdot, \phi, 0)$ exists for all finite time. The fact that $T_0(t)\phi$ remains in $\bar{B}$ was shown in section 2 and from this, it is clear that, in fact, $T_0(t)\phi$ remains in $\bar{B}_{-1}$ for all finite time.

The proof for the flow $(7.6)_n$ is similar. Integrating $(7.6)_n$ gives us

$$x(t) = (1 + x(t_0)) \exp\left\{-rT \int_{t_0}^{t} \int_{-\infty}^{0} x_s(\theta)\, Q_n(\theta)\, d\theta\, ds\right\} - 1 \qquad (7.8)_n$$

Again, if $\phi(0) > -1$ then $x(t) > -1$ as long as it is defined, and if $\phi(\theta) > -1$ for all $\theta \in (-\infty, 0]$, then

$$x(t) < (1 + \phi(0))\, e^{rT(t - t_0)} - 1. \qquad (7.10)$$

Therefore, $x(t)$ cannot become unbounded in finite time and $T_n(t)\phi := x_t(\cdot, \phi, n)$ exists for all finite time and $T_n(t)\phi$ remains in $\bar{B}_{-1}$.

Now let Γ be a bounded subset of $\bar{B}_{-1}$. To show that there is an $R > 1$, independent of n, such that $T_n(t)\phi$ eventually enters the ball $\bar{b}(R)$, we show that there is an $R_2 > 1$, independent of n, such that $x(t)$ eventually enters and remains in the ball, $b(R_2)$, with radius R_2 in $\mathbb{R}$ and then we use the inequality (7.7). To this end, we define the Lyapunov function $V(x) = (x^2/2)$. We will show that for some R_1, $1 < R_1 < R_2$, if $x(t_0) > R_1$, then for each problem $(7.8)_n$, the derivative of V along solutions satisfies $\dot{V}(x(t_0)) < -\rho < 0$. In that case, $V(x(t)) < -\rho(t - t_0) + V(x(t_0))$ as long as $x(t) > R_1$ and so $-1 < x(t) < x(t_0)$. Therefore, if $x(t_0) > R_1$, then $x(t)$ must enter $b(R_2)$ when $-\rho(t - t_0) + V(x(t_0)) = R_2{}^2$ or $t = (-R_2^2 + V(x(t_0))/\rho) + t_0$. If $x(t_0) < R_2$, then $x(t)$ must remain in $b(R_2)$ since if it leaves, then there must be some time t_1 such that $x(t_1) > R_1$, but then, for as long as $x(t)$ remains bigger than R_1, we must have $x(t) < x(t_1)$. The time that it takes to enter $b(R_2)$ can be chosen independently of ϕ since $(-R_2{}^2 + V(x(t_0))/\rho) < (-R_2^2 + C_\Gamma^2/\rho) + t_0$ where $C_\Gamma > \|\phi\|_B$ for all $\phi \in \Gamma$.

First we prove this for the flow $(7.6)_\infty$. If $x(t) > R_1$, then by $(7.8)_\infty$

$$(1 + x(t-1)) \exp\left\{-\int_{t-2}^{t-1} x(s)\, ds\right\} = x(t) + 1 > R_1 + 1$$

and so

$$x(t-1) > (R_1 + 1) \exp\left\{+\int_{t-2}^{t-1} x(s)\, ds\right\} - 1 > (R_1 + 1)\, e^{-1} - 1$$

The derivative of V along solutions of $(7.6)_\infty$ then satisfies

$$\begin{aligned} \dot{V}(x(t)) &= -x(t-1)(x(t) + x^2(t)) \\ &< -((R_1 + 1)\, e^{-1} - 1)(R_1 + R_1^2) \\ &:= -\rho \end{aligned}$$

ρ is positive as long as we choose R_1 so that $(R_1 + 1)\, e^{-1} - 1 > 0$, or $R_1 > e - 1$. We then can choose any $R_2 > R_1$ and we have proved that, for the flow $(7.6)_\infty$, $x(t)$ eventually enters and remains in $b(R_2)$.

Now we compute the derivative of V along solutions of $(7.6)_n$.

$$\dot{V}(x(t)) = -(x(t) + x^2(t)) \int_{-\infty}^{0} x(t+\theta)\, Q_n(\theta)\, d\theta$$

If for some R_1, $1 < R_1 < R_2$, $x(t) > R_1$, then from $(7.8)_n$ we have

$$(1 + x(t_0)) \exp\left\{-\int_{t_0}^{t} \int_{-\infty}^{0} x_s(\theta)\, Q_n(\theta)\, d\theta\, ds\right\} = x(t) + 1 > R_1 + 1$$

and so

$$\begin{aligned} x(t_0) &> (R_1 + 1) \exp\left\{-\int_{t_0}^{t} ds\right\} - 1 \\ &= (R_1 + 1)\, e^{-(t - t_0)} - 1. \end{aligned}$$

Therefore,

$$\inf_{t-1 \leqslant s \leqslant t} x(s) > (R_1 + 1)\, e^{-1} - 1.$$

This gives us an upper bound for $\dot{V}(x(t))$,

$$\begin{aligned}
\dot{V}(x(t)) &= -(x(t)+x^2(t))\left[\int_{-\infty}^{-1} x(t+\theta)\,Q_n(\theta)\,d\theta + \int_{-1}^{0} x(t+\theta)\,Q_n(\theta)\,d\theta\right] \\
&\leqslant -(R+R^2(t))\left[-\int_{-\infty}^{-1} Q_n(\theta)\,d\theta + (R_1+1)\,e^{-1} - 1\right] \\
&= -(R+R^2(t))\left[-\int_{-\infty}^{-1} Q_1(\theta)\,d\theta + (R_1+1)\,e^{-1} - 1\right] \\
&= -(x(t)+x^2(t))(-2e^{-1} + (R_1+1)\,e^{-1} - 1) \\
&:= -\rho
\end{aligned}$$

ρ is positive as long as we choose $R_1 > e+1$. If we choose any $R_2 > R_1$, then we have proven that, for the flow $(7.6)_n$, $x(t)$ eventually enters and remains in $b(R_2)$.

It is clear that we can choose R_2 independently of n by choosing $R_2 > e+1$ in both cases. We can then get the result if we apply inequality (7.7) in the following way. Pick Θ large enough so that $x(t) < R_2$ for all $t > \Theta$. Let $R > R_2$ and define $\eta := R - 2R_2$. Since the operator $T(\Theta)$ is bounded, we can pick $\bar{t}$ so that $(1 + \bar{t} - \Theta)\,e^{-(\bar{t}-\Theta)}\,\|T(\Theta)\|_{\bar{B}} \leqslant \eta$. Then, from the inequality, we have

$$\|x_t\|_{\bar{B}} \leqslant 2R_2 + \eta =: R$$

for all $t > \bar{t}$ and the proposition is proved.

Now we show that (c2) and (c3) hold.

PROPOSITION 7.8. *If Γ is a bounded subset of $\bar{B}_{-1}$, then the set $F(2, \Gamma) := \bigcup_{n \geqslant 0} T_n(2)\,\Gamma$ is C_2-equicontinuous.*

Proof. If $x(t)$ is a solution of $(7.6)_\infty$ with initial condition $\phi \in \bar{B}_{-1}$, then from (7.9) we have

$$|x(t)| < |1 + \phi(0)|\, e^{rTt} + 1$$

and so

$$\sup_{0 \leqslant t \leqslant 2} |x(t)| \leqslant |1 + \phi(0)|\, e^{2rT} + 1 \leqslant (1 + \|\phi\|_{\bar{B}})\, e^{2rT} + 1$$

and if $x(t)$ is a solution of $(7.6)_n$ with initial condition $\phi \in \bar{B}_{-1}$, then from (7.10) we also have

$$|x(t)| \leqslant |1 + \phi(0)|\, e^{rTt} + 1$$

and so

$$\sup_{0 \leqslant t \leqslant 2} |x(t)| \leqslant |1 + \phi(0)| \, e^{2rT} + 1 \leqslant (1 + \|\phi\|_{\bar{B}}) \, e^{2rT} + 1.$$

Therefore, $F(2, \Gamma)$ is C_2-equibounded.

To prove that $F(2, \Gamma)$ is C_2-equicontinuous, we will need to estimate the B_n-norm of the translation operator, $S(t)$, acting on $\bar{B}$. In particular, we need the following for $t \in [0, 2]$

$$\begin{aligned}
&\int_{-\infty}^{-t} |\phi(t+\theta)| \, Q_n(\theta) \, d\theta \\
&\quad = \int_{-\infty}^{0} \phi(s) \, Q_n(s-t) \, ds \\
&\quad \leqslant \sup_{-2 \leqslant \theta \leqslant 0} |\phi(s)| + \int_{-\infty}^{-2} |\phi(s)| \, Q_n(s-t) \, ds \\
&\quad \leqslant \sup_{-2 \leqslant \theta \leqslant 0} |\phi(s)| + \sup_{-\infty \leqslant s \leqslant -2} \frac{Q_n(s-t)}{Q_1(s-t)} \int_{-\infty}^{-2} |\phi(s)| \, Q_1(s-t) \, ds \\
&\quad \leqslant \sup_{-2 \leqslant \theta \leqslant 0} |\phi(s)| + (1 + 8e^{-1}) \, \bar{M}(t) \, \|\phi\|_{\bar{B}} \\
&\quad \leqslant \left(1 + (1 + 8e^{-1})(1+t) \, e^{-t} \right) \|\phi\|_{\bar{B}} \\
&\quad \leqslant (2 + 8e^{-1}) \, \|\phi\|_{\bar{B}}
\end{aligned}$$

From above, if $x(t)$ is a solution of $(7.6)_\infty$ and $t \in [0, 1]$ then

$$|\dot{x}(t)| \leqslant rT \, |x(t-1)| \, |1 + x(t)|$$

$$\begin{aligned}
\sup_{0 \leqslant t \leqslant 1} |\dot{x}(t)| &\leqslant rT \, \|\phi\|_{\bar{B}} \left(1 + (1 + \|\phi\|_{\bar{B}}) \, e^{rT} + 1 \right) \\
&= rT \, \|\phi\|_{\bar{B}} \left(2 + (1 + \|\phi\|_{\bar{B}}) \, e^{rT} \right)
\end{aligned}$$

and if $t \in [1, 2]$

$$\sup_{1 \leqslant t \leqslant 2} |\dot{x}(t)| \leqslant rT \left(1 + (1 + \|\phi\|_{\bar{B}}) \, e^{rT} \right) \left(2 + (1 + \|\phi\|_{\bar{B}}) \, e^{2rT} \right)$$

so

$$\sup_{0 \leqslant t \leqslant 2} |\dot{x}(t)| \leqslant rT \left(1 + (1 + \|\phi\|_{\bar{B}}) \, e^{rT} \right) \left(2 + (1 + \|\phi\|_{\bar{B}}) \, e^{2rT} \right)$$

If $x(t)$ is a solution of $(7)_n$, then

$$|\dot{x}(t)| \leqslant rT \int_{-\infty}^{0} |x(t+\theta)|\; Q_n(\theta)\, d\theta\, |1+x(t)|$$

$$\leqslant rT \left[\int_{-\infty}^{-t} x(t+\theta)\, Q_n(\theta)\, d\theta + \int_{-t}^{0} x(t+\theta)\, Q_n(\theta)\, d\theta \right] |1+x(t)|$$

$$\leqslant rT \left[\int_{-\infty}^{0} |\phi(s)|\; Q_n(s-t)\, ds \right.$$

$$\left. + \int_{-t}^{0} (1 + (1 + \|\phi\|_{\bar{B}})\, e^{rT(t+\theta)})\, Q_n(\theta)\, d\theta \right] (2 + (1 + \|\phi\|_{\bar{B}})\, e^{rTt})$$

$$\leqslant rT \left((2 + 8e^{-1})\, \|\phi\|_{\bar{B}} + 1 + (1 + \|\phi\|_{\bar{B}})\, e^{rTt} \right) (2 + (1 + \|\phi\|_{\bar{B}})\, e^{rTt})$$

so

$$\sup_{0 \leqslant t \leqslant 2} |\dot{x}(t)| \leqslant rT((2 + 8e^{-1})\, \|\phi\|_{\bar{B}} + 1 + (1 + \|\phi\|_{\bar{B}})\, e^{2rT})$$

$$\times (2 + (1 + \|\phi\|_{\bar{B}})\, e^{2rT}).$$

Therefore, for all $n \geqslant 0$, we have

$$\sup_{0 \leqslant t \leqslant 2} |\dot{x}(t)| \leqslant rT \left((2 + 8e^{-1})\, \|\phi\|_{\bar{B}} + 1 + (1 + \|\phi\|_{\bar{B}})\, e^{2rT} \right)$$

$$\times (2 + (1 + \|\phi\|_{\bar{B}})\, e^{2rT}).$$

and so $F(2, \Gamma)$ is C_2-equicontinuous.

PROPOSITION 7.9. *$f_n \to f$ as $n \to \infty$ uniformly on C_2-equicontinuous sets.*

Proof. We consider the difference

$$\left| \int_{-\infty}^{0} \phi(\theta)\, Q_n(\theta)\, d\theta (1 + \phi(0)) - \phi(-1)(1 + \phi(0)) \right|$$

$$\leqslant |1 + \phi(0)| \int_{-\infty}^{0} |\phi(\theta) - \phi(-1)|\; Q_n(\theta)\, d\theta$$

$$= |1 + \phi(0)| \left[\int_{-\infty}^{-1-\delta} |\phi(\theta) - \phi(-1)|\; Q_n(\theta)\, d\theta \right.$$

$$\left. + \int_{-1-\delta}^{-1+(1/n)} |\phi(\theta) - \phi(-1)|\; Q_n(\theta)\, d\theta \right]$$

$$\leqslant |1+\phi(0)| \left[\int_{-\infty}^{-1-\delta} |\phi(\theta)| \, Q_n(\theta) \, d\theta + \int_{-\infty}^{-1-\delta} |\phi(-1)| \, Q_n(\theta) \, d\theta \right.$$

$$\left. + \int_{-1-\delta}^{-1+(1/n)} |\phi(\theta)-\phi(-1)| \, Q_n(\theta) \, d\theta \right]$$

$$:= |1+\phi(0)| \, [I_1+I_2+I_3].$$

Since $F(2, \Gamma)$ is C_2-equicontinuous, for any given ε, we can find an n_1 and a δ, independent of ϕ, so that $|\phi(\theta)-\phi(-1)|<\varepsilon$ for all $\theta \in [-1-\delta, 1+(1/n)]$. Therefore,

$$I_3 < \varepsilon \qquad \text{for all} \quad n \geqslant n_1.$$

By condition (d3), given ε and δ, we can find an n_2 such that

$$I_2 \leqslant \|\phi\|_{\bar{B}} \int_{-\infty}^{-1-\delta} Q_n(\theta) \, d\theta < \varepsilon \qquad \text{for all} \quad n \geqslant n_2$$

and by Proposition 7.2, given ε and δ, we can find an n_3 such that

$$I_1 \leqslant \sup_{-\infty \leqslant \theta \leqslant -1-\delta} \frac{Q_n(\theta)}{Q(\theta)} \int_{-\infty}^{-1-\delta} |\phi(\theta)| \, Q_1(\theta) \, d\theta$$

$$\leqslant (2n^2+n) \, e^{-\delta(n+1)} \, \|\phi\|_{\bar{B}} < \varepsilon \qquad \text{for all} \quad n \geqslant n_3.$$

Therefore, for any ε we can find an $\bar{n} = \max\{n_1, n_2, n_3\}$, independent of $\phi \in \Gamma$, such that

$$\left| \int_{-\infty}^{0} \phi(\theta) \, Q_n(\theta) \, d\theta (1+\phi(0)) - \phi(-1)(1+\phi(0)) \right| < (1+C_{\Gamma}) \, 3\varepsilon$$

and so the proposition is proved.

Therefore, conditions (L), (UBD), and (c1) through (c3) are satisfied and the Theorem is proved.

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